SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)\, g(y)$ USING SEPARATION OF VARIABLES

So far, we have attempted to solve differential equations of the form $\frac{dy}{dx} = f(x)$ and also $\frac{dy}{dx} = g(y)$

We are now going to attempt to solve differential equations of the form $\frac{dy}{dx} = f(x)\, g(y)$. This is more complicated than the problems above as there are both a function of x and a function of y on the right-hand side of the equation.

The key to solve this kind of equation is the separation of the two variables onto either side of the differential equation.

In other words, transform the differential equation to: $$\frac{dy}{g(y)} = f(x)\, dx$$ and then integrate both sides, as per the examples below.

Example 19

Find the general solution of the differential equation $\frac{dy}{dx} = x(y-1)$.

Solution

The dependent and independent variables are separated onto either side of the equality, with the dependent on the left and the independent variable on the right: $\frac{1}{y-1}\frac{dy}{dx} = x$.

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx = dy$ on the LHS:

$$\int \frac{1}{y-1}\frac{dy}{dx}dx = \int x\, dx$$

$$\int \frac{1}{y-1}dy = \int x\, dx$$

$$\log_e|y-1| = \frac{1}{2}x^2 + c$$

Represent the dependent variable as an explicit function of the independent variable.

Exponentiating both sides:

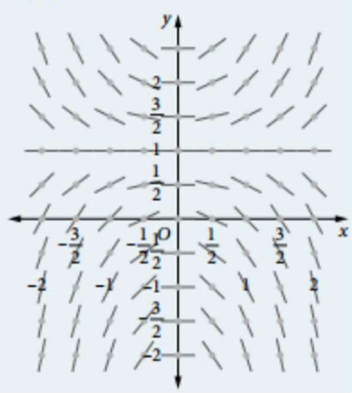
$$|y-1| = e^{\frac{1}{2}x^2+c}$$

$$= e^c e^{\frac{1}{2}x^2}$$

Removing the absolute value: $y = 1 + Ae^{\frac{1}{2}x^2}$, where $A = \pm e^c$.

It appears that the general solution of $\frac{dy}{dx} = x(y-1)$ is $y = 1 + Ae^{\frac{1}{2}x^2}$, Real A, $A \neq 0$.

The slope field of $\frac{dy}{dx} = x(y-1)$ is shown at right.



This slope field indicates that $y = 1$ should also be a solution curve for $\frac{dy}{dx} = x(y-1)$. However, the general solution $y = 1 + Ae^{\frac{1}{2}x^2}$ cannot give $y = 1$ because $A = \pm e^c$, $A \neq 0$.

In the example above, the solution $y = 1$ is an example of a so-called singular solution because it is not part of the general solution for any allowable value of the constant of integration. However, the singular solutions of $\frac{dy}{dx} = f(x)g(y)$ are often of the form $y = y^*$ for some root k of the equation $f(y^*) = 0$. The singular solutions are also usually evident from an investigation of the slope field.

Warning

To avoid missing any possible singular solutions when finding the general solution of a differential equation of the form $\frac{dy}{dx} = f(y)g(x)$, always remember to investigate the roots $\{y^* : f(y^*) = 0\}$ and/or the slope field.

Finding the popular solution of a first-order differential equation by the method of separation of variables

Example 20

Find the particular solution of Bernoulli's problem $\frac{dy}{dx} = \frac{2y}{x}$ passing through the point (1, –4).

Solution

The dependent and independent variables are separated onto either side of the equality with the dependent on the left and the independent variable on the right: $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx = dy$ on the LHS of the equation: $\int \frac{1}{y}\frac{dy}{dx}dx = 2\int \frac{1}{x}dx$

$$\int \frac{1}{y}dy = 2\int \frac{1}{x}dx$$
$$\log_e|y| = 2\log_e x + c$$
$$\log_e|y| = \log_e x^2 + c$$

Solved for the dependent variable to obtain the general solution: $\log_e|y| - \log_e x^2 = c$

$$\log_e \frac{|y|}{x^2} = c$$
$$\frac{|y|}{x^2} = e^c$$

$\therefore\ y = Ax^2$, where $A = \pm e^c$

Constant of integration is found to satisfy the required initial condition $(x, y) = (1, -4)$: $-4 = A$

Constant of integration is substituted to specify the particular solution of the problem: $y = -4x^2$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)\, g(y)$ USING SEPARATION OF VARIABLES

To solve the first-order differential equation $\frac{dy}{dx} = f(y)g(x)$ by the method of separation of variables:

1. Solve $\{y^* : f(y^*) = 0\}$ for any steady state (or equilibrium) solutions $y = y^*$.
2. Separate the dependent and independent variables onto either side of the equality $\frac{1}{f(y)}\frac{dy}{dx} = g(x)$.
3. Integrate both sides of the equality with respect to the independent variable:
   For the integral involving y terms, apply the change of variable $\int \frac{1}{f(y)}\frac{dy}{dx}dx = \int \frac{1}{f(y)}dy$.
   Add the constant of integration C to the side with the independent variable. This is the general solution.
4. Wherever possible, you should represent the dependant variable as an explicit function of the independent variable. However, you may need to be satisfied with an equation that determines the dependant variable as an implicit function of the independent variable.
5. If an initial condition $y(a) = b$ is given, solve for the constant of integration C. This is the particular solution.

You have now learnt two different methods to solve the differential equation $\frac{dy}{dx} = f(y)$.

The reciprocal method

1. Take the reciprocal of both sides of $\frac{dy}{dx} = f(y)$ to obtain $\frac{dx}{dy} = \frac{1}{f(y)}$.
2. Integrate the result with respect to y to obtain $\int \frac{dx}{dy}dy = \int \frac{1}{f(y)}dy$.

The separation of variables method

1. Separate the two variables in $\frac{dy}{dx} = f(y)$ to obtain $\frac{1}{f(y)}\frac{dy}{dx} = 1$.
2. Integrate the result with respect to x to obtain $\int \frac{1}{f(y)}\frac{dy}{dx}dx = \int dx$ or $\int \frac{1}{f(y)}dy = \int dx$.

Each of the two methods above give the original independent variable x as a function of the original dependent variable y, which must then be inverted to give the required solution $y(x)$.