PROVING DIVISIBILITY BY INDUCTION

1 If k and M are integers, which of the following expressions does *not* always generate an integer?

A $9M+4\times7k$ **B** $9M-4\times7k$ $\boxed{\textbf{C}}$ $9M\div4\times7k$ **D** $9M\times4\times7k$

Prove by induction

4 3^n+2^n is divisible by 5 for all odd integers $n\geq1$. **5** $5^n+2(11^n)$ is a multiple of 3 for all positive integers n.

4) For $n=1$ $3^1+2^1=5$ which is divisible by 5. So it's true for $n=1$

We assume it's true for $n=k$, i.e. $\exists q\in\mathbb{Z}$ such that $3^k+2^k=5q$.

In that case $3^{k+2}+2^{k+2}=3^k\times3^2+2^{k+2}$

$$=[5q-2^k]\times9+2^k\times4$$
$$=45q-2^k\times9+2^k\times4$$
$$=15q+2^k[-9+4]=15q-5\times2^k$$
$$=5[3q-2^k]$$

So it's also true for $k+2$. $\therefore$ it's true for all odd integers $n\geqslant1$

5) $n=1$ $5^1+2\times11^1=5+22=27$ which is divisible by 3

So it's true for $n=1$

Assume it's true for $n=k$, i.e. $\exists q\in\mathbb{Z}$ such that $5^k+2(11^k)=3q$.

or $5^k=3q-2\times11^k$

In that case:

$$5^{k+1}+2\times11^{k+1}=5^k\times5+2\times11^{k+1}$$
$$=[3q-2\times11^k]\times5+2\times11^{k+1}$$
$$=15q-10\times11^k+2\times11^{k+1}$$
$$=15q+11^k[-10+2\times11]$$
$$=15q+12\times11^k$$
$$=3[5q+4\times11^k]$$

So it's also true for $k+1$.

As it's true for $n=1$, it's true for any $n\geqslant0$ integers.

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6 **(a)** Factorise $k(k+1)(k+2)+3(k+1)(k+2)$. $= E$

(b) Hence prove that $n(n+1)(n+2)$ is divisible by 3 for all positive integers n.

a) $E = (k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$

b) $n(n+1)(n+2) = (n+1)(n+2)(n+3) - 3(n+1)(n+2)$.

Step 1 For $n=1$ $1 \times 2 \times 3$ is divisible by 3.

Step 2 we assume it's true for k.

i.e. $\exists\, q \in \mathbb{Z}$ such that $k(k+1)(k+2) = 3q$.

In that case $(k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$

$$\_\_\_\_\_\_ = 3q + 3(k+1)(k+2)$$

$$\_\_\_\_\_\_ = 3\left[q + (k+1)(k+2)\right]$$

$\left[q + (k+1)(k+2)\right]$ is an integer.

So $(k+1)(k+2)(k+3)$ is divisible by 3 indeed

So it's also true for $n = k+1$.

Step 3: we demonstrated it's true for $n=1$

we showed it's true for $(k+1)$ if it's true for k.

Therefore it's true for any positive integer n.

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7 $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n.

8 $7^n - 2^n$ is divisible by 9 for all even integers greater or equal to 2

Q7 Step 1 for $n=1$ $3^{3\times 1} + 2^{1+2} = 3^3 + 2^3 = 27 + 8 = 35$
which is indeed divisible by 5.

Step 2. let assume it's true for k, i.e. there exists an integer p such that $3^{3k} + 2^{k+2} = 5p$. (ie. $3^{3k} = 5p - 2^{(k+2)}$)

For k+1, $3^{3(k+1)} + 2^{(k+1)+2} = 27 \times 3^{3k} + 2^{k+3}$

$= 27\left[5p - 2^{k+2}\right] + 2^{k+3}$ (using the assumption)

$= 5 \times 27p - 27 \times 2^{k+2} + 2 \times 2^{k+2}$

$= 5 \times 27p - 2^{k+2} \times 25$

$= 5\left[27p - 5 \times 2^{k+2}\right]$

$27p - 5 \times 2^{k+2}$ is an integer, $\therefore$ $3^{3(k+1)} + 2^{(k+1)+2}$ is divisible by 5

Step 3: * true for $n=1$. * it's true for $(k+1)$ if it's true for k.
$\therefore$ by induction, it's true $\forall\, n \geq 1$

$\boxed{\text{Q8}}$ Step 1 $n=2$ $7^2 - 2^2 = 49 - 4 = 45$ which is divisible by 9

Step 2 We assume it's true for k, i.e. there exists p integer s.t. $7^k - 2^k = 9p$.

in that case $7^{k+2} - 2^{k+2} = 7^2 \times \left[9p + 2^k\right] - 2^{k+2}$

$= 7^2 \times 9p + 2^k\left[49 - 4\right]$

$= 9\left[49p + 5 \times 2^k\right]$

$49p + 5 \times 2^k$ is an integer, so indeed $7^{k+2} - 2^{k+2}$ is divisible by 9.

Step 3. True for $n=2$
true for k+2 if true for k. $\therefore$ by induction, it's true for all even integers.

PROVING DIVISIBILITY BY INDUCTION

14 **(a)** Show that $(k+3)^3 = k^3 + 9k^2 + 27k + 27$.

(b) Hence prove that the sum of the cubes of three consecutive positive integers is divisible by 3.

a) $(k+3)^3 = k^3 + 3 \times k^2 \times 3 + 3k \times 3^2 + 3^3 = k^3 + 9k^2 + 27k + 27$

b) We want to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 3

For $n=1$ $\quad 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ which is divisible by 3

Assume it's true for $n=k$, i.e. $\exists\, q \in \mathbb{Z}$ such that:

$$k^3 + (k+1)^3 + (k+2)^3 = 3q.$$

Now for $n = k+1$:

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = [3q - k^3] + (k+3)^3$$

$$= [3q - k^3] + [k^3 + 9k^2 + 27k + 27]$$

$$= 3q + 9k^2 + 27k + 27$$

$$= 3[q + 3k^2 + 9k + 9]$$

$(q + 3k^2 + 9k + 9)$ is an integer, being a sum and a product of integers.

So $(k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 3.

Step 3: we've shown it's true for $n=1$

we've shown it's true for $(k+1)$ if it's true for k.

$\therefore$ it's true for any n positive integer.

PROVING DIVISIBILITY BY INDUCTION

15 Prove that the polynomial $(x-1)^{n+2}+x^{2n+1}$ is divisible by x^2-x+1 for all positive integers n.

(*Note:* In step 2, you can't say $(x-1)^{k+2}+x^{2k+1}=(x^2-x+1)M$ where M is an integer. You must say $(x-1)^{k+2}+x^{2k+1}=(x^2-x+1)M(x)$, where $M(x)$ is a polynomial, and continue this through the rest of the proof.)

Step 1 : for $n=1$ $\quad (x-1)^{1+2}+x^{2\times1+1}=(x-1)^3+x^3$

$$= [(x-1)+x][(x-1)^2-(x-1)x+x^2]$$

as $a^3+b^3=(a+b)(a^2-ab+b^2)$

So $(x-1)^3+x^3=(2x-1)[x^2-x+1]$ so it's true for $n=1$

Step 2 : we assume it's true for $n=k$, i.e. $\exists\ M(x)$ polynomial.

$$(x-1)^{k+2}+x^{2k+1}=M(x)[x^2-x+1]$$

In that case $(x-1)^{(k+1)+2}+x^{2(k+1)+1}=(x-1)^{k+3}+x^{2k+3}$

$$= (x-1)^{k+2}(x-1)+x^{2k+3}$$

$$= \left[M(x)[x^2-x+1]-x^{2k+1}\right](x-1)+x^{2k+3}$$

$$= [x^2-x+1]M(x)(x-1)-(x-1)x^{2k+1}+x^{2k+3}$$

$$= [x^2-x+1]M(x)(x-1)+x^{2k+1}[x^2-x+1]$$

$$= [x^2-x+1]\left[M(x)(x-1)+x^{2k+1}\right]$$

$\therefore$ it's also true for $(k+1)$ if it's true for k.

Step 3 : we've shown it's true for $n=1$

we've demonstrated it's true for $(k+1)$ if it's true for k.

Therefore, by induction, it's true for all positive integers.

PROVING DIVISIBILITY BY INDUCTION

16 Prove that $x^n - 1$ is divisible by $x - 1$ for all positive integers n. (Use $x^{k+1} - 1 = x^{k+1} - x^k + x^k - 1$.)

Step 1. for $n = 1$ $x^1 - 1 = x - 1$ is indeed divisible by $(x-1)$.

Step 2: We assume it's true for $n = k$, i.e,

$\exists$ $P(x)$ polynomial such that $x^k - 1 = P(x)(x-1)$

In that case $x^{k+1} - 1 = x^{k+1} - x^k + x^k - 1$

$$= x^k(x-1) + (x-1)P(x)$$

$$= (x-1)\left[P(x) + x^k\right]$$

$\therefore$ it's also true for $k+1$

Step 3 we've demonstrated $(x^n - 1)$ is divisible by $(x-1)$ for $n = 1$

We've proved that if $(x^k - 1)$ is divisible by $(\cancel{x} - 1)$ then $(x^{k+1} - 1)$ is also divisible by $(x-1)$

Therefore it's true for all positive integers, by induction.