PROJECTILE MOTION

Projectile motion is an example of two-dimensional motion in the vertical plane under gravity. In this course, the effect of air resistance is ignored, so projectile motion is a vector combination of horizontal constant motion with vertical motion under gravity.

The assumptions made in this model are:

- the projectile is a point
- the force due to air resistance is negligible
- the only force acting on the projectile is a constant force due to gravity, assuming that the projectile is moving close to the Earth's surface.
- the acceleration due to gravity is given by the constant $g = 9.8\,\text{m s}^{-2}$, unless given otherwise.

In this vector approach the acceleration vector will be written as $\underset{\sim}{a}(t) = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$, the velocity vector as $\underset{\sim}{v}(t) = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$, and the position vector as $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$. As the only vertical acceleration is due to gravity and there is no horizontal acceleration, the acceleration vector becomes $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$.

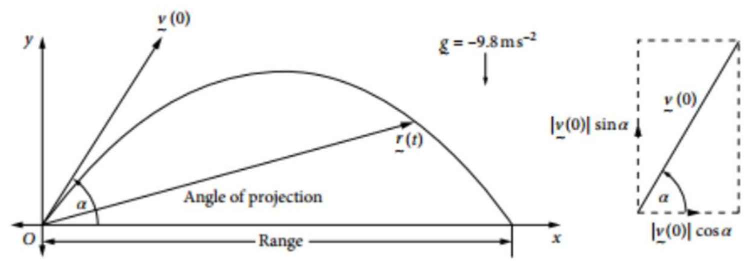
Projectile motion—a typical problem

In projectile motion, a typical problem to be considered might look like the following example:

'A particle is projected in the vertical plane from a point on horizontal ground with an initial velocity $\underset{\sim}{v}(0)\,\text{m s}^{-1}$ at an angle $\alpha°$ to the horizontal.'

In this situation, $\alpha°$ is called the angle of projection. The only force acting on the projectile is due to gravity and it acts only in the vertical plane.

It is standard to take the directions to the right and upwards as the positive directions of motion. This means that the acceleration due to gravity is negative in this case, as it is acting in the opposite direction to the motion, and so $g = -9.8\,\text{m s}^{-2}$.



A particle is projected from the point O with an initial velocity $\underset{\sim}{v}(0)$ at an angle α to the horizontal as shown. The position vector of the particle is $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$. The only force acting on the particle is the gravitational acceleration $\underset{\sim}{g}$ acting vertically downwards. When $t = 0$, $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$, while $\underset{\sim}{v}(0) = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = \underset{\sim}{0}$.

You can use integration and the initial conditions to derive the equations for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

To find velocity, integrate $\underset{\sim}{a}(t)$:

$$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$$
$$= \int -g\underset{\sim}{j}\,dt$$
$$= -gt\underset{\sim}{j} + \underset{\sim}{c}$$
$$\underset{\sim}{v}(0) = -g \times (0)\underset{\sim}{j} + \underset{\sim}{c}$$
$$= \underset{\sim}{c}$$

From initial conditions:

$$\underset{\sim}{v}(0) = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$$
$$\therefore \underset{\sim}{c} = |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$$
$$\therefore \underset{\sim}{v}(t) = -gt\underset{\sim}{j} + |\underset{\sim}{v}(0)|\cos s\alpha\underset{\sim}{i} + |\underset{\sim}{v}(0)|\sin\alpha\underset{\sim}{j}$$
$$= |\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha - gt\right)\underset{\sim}{j}$$

To find position, integrate $\underset{\sim}{v}(t)$:

$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$$
$$= \int \left(|\underset{\sim}{v}(0)|\cos\alpha\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha - gt\right)\underset{\sim}{j}\right)dt$$
$$= \left(|\underset{\sim}{v}(0)|\cos\alpha\, t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$$

From initial conditions: $\underset{\sim}{r}(0) = \underset{\sim}{0}$

$$\therefore \underset{\sim}{d} = \underset{\sim}{0}$$
$$\therefore \underset{\sim}{r}(t) = \left(|\underset{\sim}{v}(0)|\cos\alpha\, t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2\right)\underset{\sim}{j}$$

In component form, $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$ so $x = |\underset{\sim}{v}(0)|\cos\alpha\, t$ and $y = |\underset{\sim}{v}(0)|\sin\alpha t - \tfrac{1}{2}gt^2$.

Example 9

A particle is projected from a horizontal plane at an angle of 30° with speed of $100\,\text{m}\,\text{s}^{-1}$. The acceleration due to gravity is $9.8\,\text{m}\,\text{s}^{-2}$.

(a) Write $\underset{\sim}{v}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ in component form.

(c) Find the velocity and position after 4 seconds.

Solution

(a) $|\underset{\sim}{v}(0)| = 100,\ \alpha = 30°$: $\underset{\sim}{v}(0) = 100\cos 30°\,\underset{\sim}{i} + 100\sin 30°\,\underset{\sim}{j}$

$$= 50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j}$$

(b) Find $\underset{\sim}{a}(t)$: $\underset{\sim}{a}(t) = -9.8\,\underset{\sim}{j}$

Integrate $\underset{\sim}{a}(t)$ with respect to t:

$$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$$

$$= \int -9.8\,\underset{\sim}{j}\,dt$$

$$= -9.8t\,\underset{\sim}{j} + \underset{\sim}{c}$$

$\underset{\sim}{v}(0) = 50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j}$: $50\sqrt{3}\,\underset{\sim}{i} + 50\,\underset{\sim}{j} = \underset{\sim}{c}$

$$\underset{\sim}{v}(t) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}$$

Integrate $\underset{\sim}{v}(t)$ with respect to t:

$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$$

$$= \int \left(50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}\right)dt$$

$$= 50\sqrt{3}\,t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j} + \underset{\sim}{d}$$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r}(t) = 50\sqrt{3}\,t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$$

(c) $t = 4$: $\underset{\sim}{v}(t) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8t)\,\underset{\sim}{j}$

$$\underset{\sim}{v}(4) = 50\sqrt{3}\,\underset{\sim}{i} + (50 - 9.8 \times 4)\,\underset{\sim}{j}$$

$$= 50\sqrt{3}\,\underset{\sim}{i} + 10.8\,\underset{\sim}{j}$$

$$\underset{\sim}{r}(t) = 50\sqrt{3}\,t\,\underset{\sim}{i} + \left(50t - 4.9t^2\right)\underset{\sim}{j}$$

$$\underset{\sim}{r}(4) = 50\sqrt{3} \times 4\,\underset{\sim}{i} + (50 \times 4 - 4.9 \times 16)\,\underset{\sim}{j}$$

$$= 200\sqrt{3}\,\underset{\sim}{i} + 121.6\,\underset{\sim}{j}$$

Equation of the path of a projectile

Previously, it was found that $\underset{\sim}{r}(t) = \left(|\underset{\sim}{v}(0)|\cos\alpha t\right)\underset{\sim}{i} + \left(|\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$, which in terms of its components gives $x = |\underset{\sim}{v}(0)|\cos\alpha t$ and $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$.

These expressions for x and y give the parametric equation of the path (or **trajectory**) of the projectile in terms of t. Combining these equations to eliminate t gives the Cartesian equation of the path of the projectile.

$x = |\underset{\sim}{v}(0)|\cos\alpha t$ gives $t = \dfrac{x}{|\underset{\sim}{v}(0)|\cos\alpha}$

Substitute this into $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$:

$$y = |\underset{\sim}{v}(0)|\sin\alpha \times \frac{x}{|\underset{\sim}{v}(0)|\cos\alpha} - \frac{1}{2}g\left(\frac{x}{|\underset{\sim}{v}(0)|\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{g}{2} \times \frac{x^2}{|\underset{\sim}{v}(0)|^2\cos^2\alpha}$$

The equation of the trajectory is $y = x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}$

This trajectory is a concave down parabola and the properties of the parabola may be used to answer questions about this path.

Greatest height of the projectile

A particle reaches its greatest height when it stops rising, which is when vertical component of $\underset{\sim}{v}(t)$ is zero.

Hence the greatest height occurs where $|\underset{\sim}{v}(0)|\sin\alpha - gt = 0$: $\quad t = \dfrac{|\underset{\sim}{v}(0)|\sin\alpha}{g}$

To find this greatest height, you can substitute this value of t into the vertical component of $\underset{\sim}{r}(t)$.

Vertical component: $y = |\underset{\sim}{v}(0)|\sin\alpha t - \frac{1}{2}gt^2$

Greatest height:
$$y = |\underset{\sim}{v}(0)|\sin\alpha \times \frac{|\underset{\sim}{v}(0)|\sin\alpha}{g} - \frac{1}{2}g\left(\frac{|\underset{\sim}{v}(0)|\sin\alpha}{g}\right)^2$$
$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g} - \frac{1}{2}\frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g}$$
$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$

Alternatively, this result could have been found using the Cartesian equation of the trajectory.

A parabola is symmetric about its turning point, so the greatest value of a concave down parabola occurs at its turning point. The abscissa (horizontal coordinate) of the turning point is the average of the abscissae of the points of intersection with the x-axis.

Substituting $y = 0$ into $y = x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}$: $\quad x\tan\alpha - \dfrac{gx^2}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} = 0$

$$x\left(\tan\alpha - \frac{gx}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha}\right) = 0$$
$$x = 0 \text{ or } x = \frac{2|\underset{\sim}{v}(0)|^2\cos^2\alpha\tan\alpha}{g}$$
$$= \frac{2|\underset{\sim}{v}(0)|^2\sin\alpha\cos\alpha}{g}$$
$$= \frac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{g}$$

$\therefore$ Greatest height occurs at $x = \dfrac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g}$

Greatest height is:
$$y = \frac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g} \times \tan\alpha - \frac{g}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} \times \left(\frac{|\underset{\sim}{v}(0)|^2\sin 2\alpha}{2g}\right)^2$$
$$= \frac{|\underset{\sim}{v}(0)|^2\, 2\sin\alpha\cos\alpha}{2g} \times \frac{\sin\alpha}{\cos\alpha} - \frac{g}{2|\underset{\sim}{v}(0)|^2\cos^2\alpha} \times \left(\frac{|\underset{\sim}{v}(0)|^4 \times 4\sin^2\alpha\cos^2\alpha}{4g^2}\right)$$
$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{g} - \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$
$$= \frac{|\underset{\sim}{v}(0)|^2\sin^2\alpha}{2g}$$

Time of flight

In the typical case, the time of flight is the time taken for the projectile to return to the ground.

This is when the vertical component of $\underset{\sim}{r}(t)$ is equal to zero.

$$y = \left|\underset{\sim}{v}(0)\right| \sin\alpha t - \frac{1}{2} g t^2$$

$$\left|\underset{\sim}{v}(0)\right| \sin\alpha t - \frac{1}{2} g t^2 = 0$$

$$t\left(\left|\underset{\sim}{v}(0)\right| \sin\alpha - \frac{1}{2} g t\right) = 0$$

$$t = 0 \text{ or } t = \frac{2\left|\underset{\sim}{v}(0)\right| \sin\alpha}{g}$$

$\therefore$ Projectile returns to the ground when $t = \dfrac{2\left|\underset{\sim}{v}(0)\right| \sin\alpha}{g}$.

Range of flight

The range is the horizontal distance from the point of projection to the point where the particle hits the ground.

It is the value of x, the horizontal component of $\underset{\sim}{r}(t)$ when $t = \dfrac{2\left|\underset{\sim}{v}(0)\right| \sin\alpha}{g}$.

$$t = \frac{2\left|\underset{\sim}{v}(0)\right| \sin\alpha}{g}: \qquad \begin{aligned} x &= \left|\underset{\sim}{v}(0)\right| \cos\alpha t \\ &= \left|\underset{\sim}{v}(0)\right| \cos\alpha \times \frac{2\left|\underset{\sim}{v}(0)\right| \sin\alpha}{g} \\ &= \frac{\left|\underset{\sim}{v}(0)\right|^2 \sin 2\alpha}{g} \end{aligned}$$

$\therefore$ Range: $R = \dfrac{\left|\underset{\sim}{v}(0)\right|^2 \sin 2\alpha}{g}$

This result is also implied when finding the greatest height using the equation of the trajectory. It is double the horizontal distance to the greatest height, which makes sense when you consider that the trajectory is a parabola.

Maximum range

For a given speed of projection $|\underset{\sim}{v}(0)|$, the equation $R = \frac{|\underset{\sim}{v}(0)|^2 \sin 2\alpha}{g}$ defines the range R as a function of the angle of projection α. This function has its greatest value where $\sin 2\alpha = 1$ or $\alpha = 45°$.

Thus the greatest range of the particle occurs where $\alpha = 45°$ and is given by: $R_{\max} = \frac{|\underset{\sim}{v}(0)|^2}{g}$

Example 10

In Example 9, you found that $\underset{\sim}{v}(0) = 50\sqrt{3}\underset{\sim}{i} + 50\underset{\sim}{j}$, $\underset{\sim}{v}(t) = 50\sqrt{3}\underset{\sim}{i} + (50 - 9.8t)\underset{\sim}{j}$ and $\underset{\sim}{r}(t) = 50\sqrt{3}t\underset{\sim}{i} + (50t - 4.9t^2)\underset{\sim}{j}$.
Use this information to find:

(a) the equation of the trajectory

(b) the greatest height reached

(c) the time of flight and the range of the flight.

Solution

(a) $\underset{\sim}{r}(t) = 50\sqrt{3}t\underset{\sim}{i} + (50t - 4.9t^2)\underset{\sim}{j}$: $x = 50\sqrt{3}t$, $y = 50t - 4.9t^2$

Hence $t = \frac{x}{50\sqrt{3}}$ so
$$y = 50 \times \frac{x}{50\sqrt{3}} - 4.9 \times \left(\frac{x}{50\sqrt{3}}\right)^2$$
$$= \frac{x}{\sqrt{3}} - \frac{4.9x^2}{7500}$$
$$= \frac{x\sqrt{3}}{3} - \frac{49x^2}{75000}$$

(b) $\underset{\sim}{v}(t) = 50\sqrt{3}\underset{\sim}{i} + (50 - 9.8t)\underset{\sim}{j}$: Greatest height occurs when $50 - 9.8t = 0$
$$t = \frac{50}{9.8} = \frac{250}{49}\text{ s}$$

$y = 50t - 4.9t^2$: Greatest height is
$$y_{\max} = 50 \times \frac{250}{49} - 4.9 \times \left(\frac{250}{49}\right)^2$$
$$= \frac{12500}{49} - \frac{4.9 \times 250^2}{49^2}$$
$$= 127.55\text{ m}$$

(c) $y = 50t - 4.9t^2$: Time of flight when $50t - 4.9t^2 = 0$
$$t(50 - 4.9t) = 0$$
$$t = 0 \text{ or } t = \frac{50}{4.9}$$

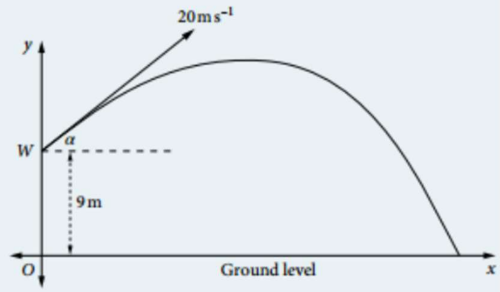
Time of flight is 10.2 s.
For the range, substitute $t = \frac{50}{4.9}$ into $x = 50\sqrt{3}t$: Range $= 50\sqrt{3} \times \frac{50}{4.9} = 883.7\text{ m}$

You can see from the calculations above why you are often given $g = 10\text{ m s}^{-2}$ rather than $g = 9.8\text{ m s}^{-2}$.

A particle is not always projected from level ground, so you will usually need to draw a diagram to show all the given information and so you can form the appropriate vector for $\underset{\sim}{a}(t)$ and the initial conditions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

Example 11

A particle is projected from a window 9 metres above the horizontal ground at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, with an initial velocity of $20\,\text{m s}^{-1}$. Use $g = 10\,\text{m s}^{-2}$.



(a) Write $\underset{\sim}{v}(0)$ and $\underset{\sim}{r}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) When does the particle hit the ground?

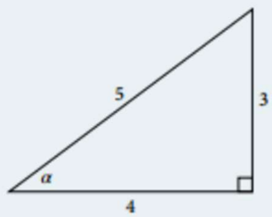
(d) What is the horizontal distance from the base of the building to where the particle hits the ground?

(e) What is the maximum height of the particle above the ground?

(f) What is the Cartesian equation of the path of the particle?

Solution

(a) $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $|\underset{\sim}{v}(0)| = 20$:



$$\underset{\sim}{v}(0) = 20\cos\alpha\underset{\sim}{i} + 20\sin\alpha\underset{\sim}{j}$$
$$= 20\times\frac{4}{5}\underset{\sim}{i} + 20\times\frac{3}{5}\underset{\sim}{j}$$
$$= 16\underset{\sim}{i} + 12\underset{\sim}{j}$$

$t = 0, x = 0, y = 9$: $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 9\underset{\sim}{j}$

(b) $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$
$$= -10\underset{\sim}{j}$$

Integrate $\underset{\sim}{a}(t)$ with respect to t: $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$
$$= -\int 10\underset{\sim}{j}\,dt$$
$$= \underset{\sim}{c} - 10t\underset{\sim}{j}$$

$\underset{\sim}{v}(0) = 16\underset{\sim}{i} + 12\underset{\sim}{j}$: $16\underset{\sim}{i} + 12\underset{\sim}{j} = \underset{\sim}{c}$

$$\underset{\sim}{v}(t) = 16\underset{\sim}{i} + 12\underset{\sim}{j} - 10t\underset{\sim}{j}$$
$$= 16\underset{\sim}{i} + (12 - 10t)\underset{\sim}{j}$$

Integrate $\underset{\sim}{v}(t)$ with respect to t:

$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$$
$$= \int \left(16\underset{\sim}{i} + (12 - 10t)\underset{\sim}{j}\right)dt$$
$$= 16t\underset{\sim}{i} + \left(12t - 5t^2\right)\underset{\sim}{j} + \underset{\sim}{d}$$

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 9\underset{\sim}{j}$: $9\underset{\sim}{j} = \underset{\sim}{d}$

$$\underset{\sim}{r}(t) = 16t\underset{\sim}{i} + \left(12t - 5t^2\right)\underset{\sim}{j} + 9\underset{\sim}{j}$$
$$= 16t\underset{\sim}{i} + \left(9 + 12t - 5t^2\right)\underset{\sim}{j}$$

(c) The particle hits the ground where $y = 0$:

$$9 + 12t - 5t^2 = 0$$
$$(3 + 5t)(3 - t) = 0$$
$$t = -\frac{3}{5} \quad \text{or} \quad t = 3$$

As $t \geq 0$, $t = 3$ and the particle hits the ground after 3 seconds.

(d) Find x when $t = 3$: $x = 16\times 3 = 48\,\text{m}$

The particle hits the ground 48 metres from the base of the building.

(e) The maximum height occurs when the vertical velocity is zero: $12 - 10t = 0$

$t = 1.2\,\text{s}$

Find y when $t = 1.2$:

$y = 9 + 12\times 1.2 - 5\times 1.2^2 = 16.2\,\text{m}$

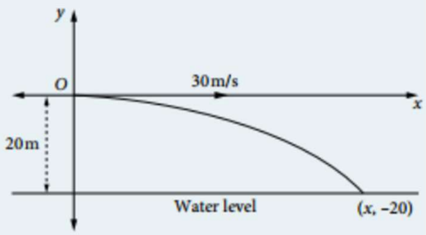
(f) $\underset{\sim}{r}(t) = 16t\underset{\sim}{i} + \left(9 + 12t - 5t^2\right)\underset{\sim}{j}$:

$x = 16t$, $y = 9 + 12t - 5t^2$

$$t = \frac{x}{16}: y = 9 + 12\times\frac{x}{16} - 5\times\left(\frac{x}{16}\right)^2$$
$$y = 9 + \frac{3x}{4} - \frac{5x^2}{256}$$

Example 12

A particle is projected horizontally from the top of a cliff 20 m above the water, with a velocity of $30\,\text{m s}^{-1}$. Use $g = 10\,\text{m s}^{-2}$.



(a) Write $\underset{\sim}{v}(0)$ and $\underset{\sim}{r}(0)$ in component form.

(b) Find $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) When does the particle hit the water?

(d) How far from the base of the cliff does the particle hit the water?

Solution

(a) Horizontal velocity, so $\underset{\sim}{v}(0) = 30\underset{\sim}{i} + 0\underset{\sim}{j}$.

The top of cliff is taken as the origin, so $\underset{\sim}{r}(0) = \underset{\sim}{0}$

The particle hits the water when $y = -20$ m.

(b) $\underset{\sim}{a}(t) = -g\underset{\sim}{j}$

Integrate $\underset{\sim}{a}(t)$ with respect to t: $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$

$$= \int -g\underset{\sim}{j}\,dt$$

$$= -gt\underset{\sim}{j} + \underset{\sim}{c}$$

$\underset{\sim}{v}(0) = 30\underset{\sim}{i} + 0\underset{\sim}{j}$: $30\underset{\sim}{i} = \underset{\sim}{c}$

$\underset{\sim}{v}(t) = 30\underset{\sim}{i} - gt\underset{\sim}{j}$

Integrate $\underset{\sim}{v}(t)$ with respect to t: $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$

$$= \int \left(30\underset{\sim}{i} - gt\underset{\sim}{j}\right)dt$$

$$= 30t\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j} + \underset{\sim}{d}$$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = 30t\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j}$

(c) $y = -20$: $-\frac{1}{2}gt^2 = -20$

$$t^2 = \frac{40}{g}$$

$g = 10$: $t^2 = \frac{40}{10} = 4$

$t = 2$ as $t \geq 0$

The particle hits the water after 2 seconds.

(d) Find x when $t = 2$:
$x = 30t = 30 \times 2 = 60$ m
The particle hits the water 60 metres from the base of the cliff.

Note: If there is no vertical component of projection, then the velocity of projection does not affect the time taken to reach the water. If you drop an object and at the same time throw another object horizontally, you will notice that both objects reach the ground at the same time (assuming that air resistance is not significantly different for the objects). In both cases the vertical motion is governed by $y = -\frac{1}{2}gt^2$, independent of any horizontal motion, just as in the previous example.

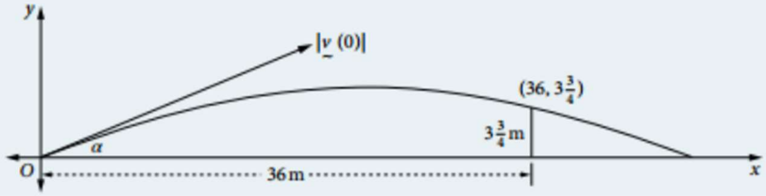
Example 13

A particle is projected from ground level and 1.5 seconds later it just clears a wall 3.75 metres high at a horizontal distance of 36 metres. If the initial velocity is $|\underline{v}(0)| = |\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}|\,\text{m s}^{-1}$ and $\underline{r}(t) = |\underline{v}(0)|\,t\cos\alpha\,\underline{i} + \left(|\underline{v}(0)|\,t\sin\alpha - 5t^2\right)\underline{j}$:

(a) Calculate the initial velocity and the angle of projection.

(b) Find expressions for $\underline{r}(t)$ and $\underline{v}(t)$ in component form.

(c) Find the range.



Solution

(a) $t = 1.5, x = 36, y = 3.75$: $\underline{r}(t) = |\underline{v}(0)|\,t\cos\alpha\,\underline{i} + \left(|\underline{v}(0)|\,t\sin\alpha - 5t^2\right)\underline{j}$

Hence $x = |\underline{v}(0)|\,t\cos\alpha$ and $y = |\underline{v}(0)|\,t\sin\alpha - 5t^2$

Solve simultaneously to find the angle of projection α.

$t = 1.5, x = 36$: $36 = 1.5|\underline{v}(0)|\cos\alpha$ [1]

$t = 1.5, y = 3.75$: $3.75 = 1.5|\underline{v}(0)|\sin\alpha - 5\times1.5^2$

$15 = 1.5|\underline{v}(0)|\sin\alpha$ [2]

[2] ÷ [1]: $\dfrac{15}{36} = \dfrac{1.5|\underline{v}(0)|\sin\alpha}{1.5|\underline{v}(0)|\cos\alpha}$

$\tan\alpha = \dfrac{5}{12}$

$\alpha = 22°37'$

To calculate the velocity of projection $|v(0)|$ substitute into [1]: $\cos(\alpha) = \dfrac{12}{13}$, as $\tan\alpha = \dfrac{5}{12}$

$\dfrac{36}{1.5} = |\underline{v}(0)| \times \dfrac{12}{13}$

$|\underline{v}(0)| = \dfrac{360}{15} \times \dfrac{13}{12} = 26\,\text{m s}^{-1}$

(b) $\underline{r}(t) = 26t\cos\alpha\,\underline{i} + \left(26t\sin\alpha - 5t^2\right)\underline{j}$

$= 26t \times \dfrac{12}{13}\underline{i} + \left(26t \times \dfrac{5}{13} - 5t^2\right)\underline{j}$

$= 24t\,\underline{i} + \left(10t - 5t^2\right)\underline{j}$

Differentiate $\underline{r}(t)$ with respect to t:

$\underline{v}(t) = 24\underline{i} + (10 - 10t)\,\underline{j}$

(c) $y = 0$: $10t - 5t^2 = 0$

$5t(2 - t) = 0$

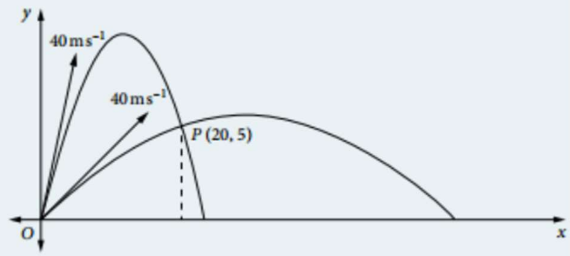
$t = 2\,\text{s}$

$x = 24 \times 2 = 48\,\text{m}$

The range is 48 metres.

Example 14

A stone is thrown to hit a small object sitting on top of a wall that is 20 metres horizontally from the point of projection and 5 metres high. If the stone is thrown from ground level with a speed of 40 m s^{-1}, show that there are two angles of projection that will allow the object to be hit.



The position vector of the particle is given by $\underset{\sim}{r}(t)=|\underset{\sim}{v}(0)|t\cos\alpha\underset{\sim}{i}+\left(|\underset{\sim}{v}(0)|t\sin\alpha-5t^2\right)\underset{\sim}{j}$ where $|\underset{\sim}{v}(0)|$ is the initial velocity and α the angle of projection from the horizontal.

Solution

$|\underset{\sim}{v}(0)|=40$, $\underset{\sim}{r}(t)=40t\cos\alpha\underset{\sim}{i}+\left(40t\sin\alpha-5t^2\right)\underset{\sim}{j}$, $x=20$, $y=5$.

$x=20$: $40t\cos\alpha=20$

$$t=\frac{1}{2\cos\alpha}$$

$y=5$: $40t\sin\alpha-5t^2=5$ [1]

Substitute $t=\dfrac{1}{2\cos\alpha}$ into [1]: $40\sin\alpha\times\dfrac{1}{2\cos\alpha}-5\left(\dfrac{1}{2\cos\alpha}\right)^2=5$

$$20\tan\alpha-\frac{5}{4\cos^2\alpha}=5$$

$$80\tan\alpha-5\sec^2\alpha=20 \qquad [2]$$

Substitute the identity $\sec^2\alpha=1+\tan^2\alpha$ into [2] to solve for α: $80\tan\alpha-5\left(1+\tan^2\alpha\right)=20$

$$5\tan^2\alpha-80\tan\alpha+25=0$$

$$\tan^2\alpha-16\tan\alpha+5=0$$

Quadratic formula: $\tan\alpha=\dfrac{16\pm\sqrt{256-20}}{2}$

$$=\frac{16\pm\sqrt{236}}{2}$$

$$=8\pm\sqrt{59}$$

$$\tan\alpha=15.68 \text{ or } 0.3189$$

$$\alpha=86°21' \text{ or } 17°41'$$

There are two angles of projection that allow the object to be hit, 17° 41′ and 86° 21′.