VELOCITY AND ACCELERATION AS A RATE OF CHANGE

1 A particle moves in a straight line so that its displacement x m from a fixed point O on the line at any time t seconds ($t \geq 0$) is given by $x = t^2 - 5t + 6$. Find:

(a) its initial displacement **(b)** its initial velocity
(c) when it first passes through O and with what velocity
(d) when it passes through O the second time and with what velocity
(e) when and where its velocity is zero.

a) At $t=0$ $x(0) = 0^2 - 5\times 0 + 6 = 6$ m

b) $\dot{x} = 2t - 5$ so $\dot{x}(0) = 2\times 0 - 5 = -5$ ms^{-1}

c) $x(t) = 0$ when $t^2 - 5t + 6 = 0$ $\Delta = 25 - 4\times 6 = 1$

So two roots $x_1 = \frac{5-1}{2} = \frac{4}{2} = 2$ and $x_2 = \frac{5+1}{2} = \frac{6}{2} = 3$.

So it first passes through O at $t = 2$ s.

At $t = 2$ s $\dot{x}(2) = 2\times 2 - 5 = 4 - 5 = -1$ ms^{-1}

d) it passes through O for the 2nd time at $t = 3$ s.

At $t = 3$ s $\dot{x}(3) = 2\times 3 - 5 = 6 - 5 = 1$ ms^{-1}

e) its velocity is zero when $2t - 5 = 0$

i.e. when $t = 2.5$ s.

At $t = 2.5$ s, $x(2.5) = 2.5^2 - 5\times 2.5 + 6$

$x(2.5) = -0.25$ m

2 The displacement x m at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = 2t^3 - t^2 + 4t + 1$. Its acceleration is given by:

A $a = 2t^3 - t^2 + 4t + 1$ **B** $a = 6t^2 - 2t + 4$ **C** $a = 12t - 2$ **D** $a = 12$

$\dot{x}(t) = 6t^2 - 2t + 4$

$\ddot{x}(t) = 12t - 2$ so $\boxed{C}$

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3 The displacement x m at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 5t + 4$.

(a) At what time is its velocity zero? **(b)** What is the acceleration at this time?

(c) What is the distance travelled in the first 4 seconds? **(d)** At what time is the velocity $8\ \text{m s}^{-1}$?

a) $\dot{x}(t) = 2t - 5$ so $\dot{x}(t) = 0$ when $t = 5/2$ s.

b) $\ddot{x}(t) = 2$ so at $t = 5/2$ s, $\ddot{x}(2.5) = 2\ \text{ms}^{-2}$

(the acceleration is constant, doesn't depend of t)

c) From $t = 0$ to $t = 2.5$:

$x(0) = 0^2 - 5 \times 0 + 4 = 4$

$x(2.5) = 2.5^2 - 5 \times 2.5 + 4$

$x(2.5) = -2.25$

so the distance between these two times is 6.25 m.

Then from $t = 2.5$ to $t = 4$:

$x(4) = 4^2 - 5 \times 4 + 4 = 0$ so distance between $t = 2.5$ and $t = 4$ is 2.25 m.

Total distance: $6.25 + 2.25 = 8.5$ m

d) $\dot{x}(t) = 8$ when $2t - 5 = 8$ i.e. $2t = 13$

so at $t = 6.5$ s.

4 A point moving in a straight line is distant x m from the origin O at time t, where $x = 2t^3 - 15t^2 + 36t$.

(a) Find the velocity and acceleration at any time t.

(b) Find the initial velocity and acceleration. **(c)** At what times is the velocity zero?

(d) At what time is the acceleration zero? Find the velocity and position at this time.

(e) During what interval of time is the velocity negative?

a) $\dot{x}(t) = 6t^2 - 30t + 36$

$\ddot{x}(t) = 12t - 30$

b) $\dot{x}(0) = 6 \times 0^2 - 30 \times 0 + 36 = 36 \text{ ms}^{-1}$

$\ddot{x}(0) = 12 \times 0 - 30 = -30 \text{ ms}^{-2}$

c) $\dot{x}(t) = 0$ when $6t^2 - 30t + 36 = 0$

$\Leftrightarrow t^2 - 5t + 6 = 0 \Leftrightarrow t = 2$ or $t = 3$

d) $\ddot{x}(t) = 0$ when $12t - 30 = 0$ i.e $t = \frac{30}{12} = 2.5 \text{ s}$

$x(2.5) = 2 \times (2.5)^3 - 15 \times (2.5)^2 + 36 \times 2.5 = 27.5 \text{m}$

$\dot{x}(2.5) = 6 \times (2.5)^2 - 30 \times 2.5 + 36 = -1.5 \text{ ms}^{-1}$

e) $\dot{x}(t) = 0$ when $6t^2 - 30t + 36 = 0$ i.e. $t = 2$ or $t = 3$

$\dot{x}(t)$ is a quadratic function, concave up, therefore is negative between these 2 values.

$\dot{x}(t) < 0$ when $2 < t < 3$

5 The displacement x m at time t seconds of a particle moving in a straight line is given by $x = 2t^3 - 9t^2 + 12t + 6$. Find:

(a) when its acceleration is zero, and its velocity at this time

(b) when its velocity is zero, and its acceleration at this time.

a) $\dot{x}(t) = 6t^2 - 18t + 12$ $\qquad$ $\ddot{x}(t) = 12t - 18$.

So $\ddot{x}(t) = 0$ when $t = \frac{18}{12} = 1.5\,s$.

$\dot{x}(1.5) = 6 \times 1.5^2 - 18 \times 1.5 + 12 = -1.5\,ms^{-1}$

b) $\dot{x}(t) = 0$ when $6t^2 - 18t + 12 = 0 \Leftrightarrow t^2 - 3t + 2 = 0$

so $t = 1$ or $t = 2$

$\ddot{x}(1) = 12 \times 1 - 18 = 12 - 18 = -6\,ms^{-2}$

whereas $\ddot{x}(2) = 12 \times 2 - 18 = 24 - 18 = 6\,ms^{-2}$

6 Two bodies move along a straight path, starting at the same time, so that their displacement x m from a fixed point O at any time t is given by $x_1 = t + 6$ and $x_2 = t^2 + 4$ respectively. At what times are they:

(a) together $\qquad$ **(b)** travelling with the same velocity?

a) $x_1(t) = x_2(t)$ when $t + 6 = t^2 + 4$

$\Leftrightarrow t^2 - t - 2 = 0$ so $t = -1$ (impossible) or $t = 2$

They are together at $t = 2\,s$.

b) $\dot{x}_1(t) = 1$ $\qquad$ $\dot{x}_2(t) = 2t$

So they have the same velocity when $1 = 2t$

i.e. when $t = 0.5\,s$

8 Two cars A and B travel along a straight road in the same direction. Their respective distances x km from a fixed point O at any time t hours are given by the following rules:

A: $x_A = 50t - 20t^2$ B: $x_B = 80t^2 + 20t$

(a) Calculate each car's speed at the point O.

(b) At what time are the cars travelling at the same speed?

(c) Both cars reach a point Q at the same time. Calculate the distance from O to Q.

(d) A third car, travelling at uniform speed, is 2 km ahead of A and B when they pass the point O. If this car arrives at Q at the same time as A and B, find a rule connecting x and t for it.

a) $x_A(t) = 50t - 20t^2$ so $\dot{x}_A(t) = 50 - 40t$

so $\dot{x}_A(0) = 50 \text{ km hr}^{-1}$

$x_B(t) = 80t^2 + 20t$ so $\dot{x}_B(t) = 160t + 20$

so $\dot{x}_B(0) = 20 \text{ km/hr}$

b) $\dot{x}_A(t) = \dot{x}_B(t)$ when $50 - 40t = 160t + 20$

i.e. when $200t = 30$ so at $t = 3/20 = 0.15$ hour (or 9 minutes)

c) $x_A(t) = x_B(t)$ when $50t - 20t^2 = 80t^2 + 20t$

i.e. when $50 - 20t = 80t + 20 \Leftrightarrow 100t = 30$

so at $t = 0.3$ hour

At $t = 0.3$, $x_A(0.3) = 50 \times 0.3 - 20 \times 0.3^2 = 13.2$ km

which is the distance from O to Q.

d) $\dot{x}_c(t) = k$ where k is a constant.

So $x_c(t)$ must be of the form $x_c(t) = kt + K$

where k, K are two constants.

At $t = 0$, $x_c(0) = 2$ so $K = 2$

At $t = 0.3$ $x_c(0.3) = 13.2$ so $k \times 0.3 + 2 = 13.2$

so $k = \frac{11.2}{0.3} = \frac{112}{3}$ so $x_c(t) = \frac{112}{3}t + 2$

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9 A particle moves in a straight line so that its displacement $x(t)$ from a fixed point in the line at time $t \geq 0$ is given by $x(t) = 3 + 4t - 5\sqrt{t^2+4}$. Find the particle's displacement when it comes to rest.

$\dot{x}(t) = 4 - 5 \times \frac{1}{2}(t^2+4)^{\frac{1}{2}-1} \times 2t$ ("Chain rule")

$\dot{x}(t) = 4 - \dfrac{5t}{\sqrt{t^2+4}} = \dfrac{4\sqrt{t^2+4} - 5t}{\sqrt{t^2+4}}$

So $\dot{x}(t) = 0$ (at rest) when $4\sqrt{t^2+4} - 5t = 0$

i.e. when $\sqrt{t^2+4} = \dfrac{5t}{4}$

$\Rightarrow t^2 + 4 = \left(\dfrac{5t}{4}\right)^2 = \dfrac{25t^2}{16}$ so $t^2\left(1 - \dfrac{25}{16}\right) = -4$

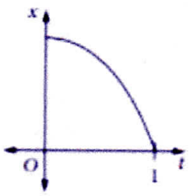
so $t^2 = \dfrac{4 \times 16}{9} = \dfrac{64}{9}$ so $t = \sqrt{\dfrac{64}{9}} = \dfrac{8}{3} = 2\frac{2}{3}$ second.

At $t = 2\frac{2}{3}$ $x\left(2\frac{2}{3}\right) = 3 + 4 \times \dfrac{8}{3} - 5\sqrt{\dfrac{64}{9} + 4}$

$x\left(\dfrac{8}{3}\right) = 3 + \dfrac{32}{3} - 5\sqrt{\dfrac{100}{9}} = \dfrac{41}{3} - \dfrac{5 \times 10}{3} = \dfrac{-9}{3} = -3$

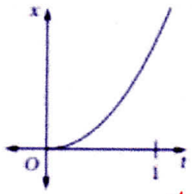
10 A particle is moving so that, for $0 < t < 1$, its velocity is positive and its acceleration is negative. Which graph could represent the displacement function of this particle?

A



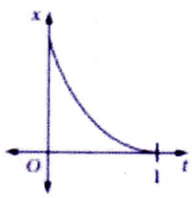
gradient negative so $\dot{x} < 0$
concave down so $\ddot{x} < 0$

B



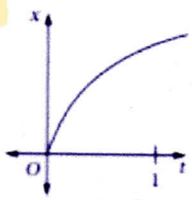
gradient positive so $\dot{x} > 0$
concave up so $\ddot{x} > 0$

C



gradient negative so $\dot{x} < 0$
concave up so $\ddot{x} > 0$

$\boxed{\text{D}}$



gradient positive so $\dot{x} > 0$
concave down so $\ddot{x} < 0$