# HALF-ANGLE FORMULAE – THE t FORMULAE

Remember from the double-angle formulae:

$tan(2\alpha) = \frac{2\,tan\,\alpha}{1-tan^2\,\alpha}$ therefore $tan\,\alpha = \frac{2\,ta\,\left(\frac{\alpha}{2}\right)}{1-tan^2\left(\frac{\alpha}{2}\right)}$

substituting $t = tan\left(\frac{\alpha}{2}\right)$ we get:

$$\boldsymbol{tan\,\alpha = \frac{2t}{1-t^2}}$$

$sin(2\alpha) = 2\,sin\,\alpha\,cos\,\alpha$ therefore $sin\,\alpha = 2\,sin\left(\frac{\alpha}{2}\right)cos\left(\frac{\alpha}{2}\right)$

$$sin\,\alpha = 2\frac{sin\left(\frac{\alpha}{2}\right)}{cos\left(\frac{\alpha}{2}\right)}cos^2\left(\frac{\alpha}{2}\right)$$

$$sin\,\alpha = 2\,tan\left(\frac{\alpha}{2}\right) \times cos^2\left(\frac{\alpha}{2}\right)$$

$$sin\,\alpha = 2\;\;tan\left(\frac{\alpha}{2}\right) \times \frac{1}{sec^2\left(\frac{\alpha}{2}\right)}$$

but (as $sec^2\,\theta = 1 + tan^2\,\theta$) so:

$$sin\,\alpha = 2\;\;tan\left(\frac{\alpha}{2}\right)\frac{1}{1+tan^2\left(\frac{\alpha}{2}\right)}$$

$$\boldsymbol{sin\,\alpha = \frac{2t}{1+t^2}}$$

For cos $\alpha$, we remember that: $tan\,\alpha = \frac{sin\,\alpha}{cos\,\alpha}$ therefore: $cos\,\alpha = \frac{sin\,\alpha}{tan\,\alpha}$

so:

$$cos\,\alpha = \frac{\frac{2t}{1+t^2}}{\frac{2t}{1-t^2}}$$

$$cos\,\alpha = \left(\frac{2t}{1+t^2}\right) \div \left(\frac{2t}{1-t^2}\right)$$

$$cos\,\alpha = \left(\frac{2t}{1+t^2}\right) \times \left(\frac{1-t^2}{2t}\right)$$

$$\boldsymbol{cos\,\alpha = \frac{1-t^2}{1+t^2}}$$

## Example 8

Given that $\tan A = \frac{5}{12}$, find the exact values of $\sin 2A$ and $\cos 2A$.

### Solution

$\tan A = \frac{5}{12}$ so $t = \frac{5}{12}$:

$$\sin 2A = \frac{2t}{1+t^2} = \frac{2 \times \frac{5}{12}}{1+\left(\frac{5}{12}\right)^2} = \frac{120}{169}$$

$$\cos 2A = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{5}{12}\right)^2}{1+\left(\frac{5}{12}\right)^2} = \frac{119}{169}$$

*Note:* If a calculator is used to find intermediate values then the answer will be only approximate.

For example, $\tan A = \frac{5}{12}$ so $A = 22°\,37'$, $2A = 45°\,14'$, $\sin 2A \approx 0.709\,98$, while $\frac{120}{169} \approx 0.710\,06$ (both to five d.p.)—close, but not the same.

## Example 9

Use the *t* formulae to prove that: $\dfrac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} = \tan\dfrac{\theta}{2}$

### Solution

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} \\
&= \frac{\left(\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1\right)}{\left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + 1\right)} \\
&= \left(\frac{1-t^2+2t-(1+t^2)}{1+t^2}\right) \div \left(\frac{1-t^2-2t+(1+t^2)}{1+t^2}\right) \\
&= \frac{1-t^2+2t-1-t^2}{1-t^2-2t+1+t^2} \\
&= \frac{2t-2t^2}{2-2t} \\
&= \frac{2t(1-t)}{2(1-t)} \\
&= t = \tan\frac{\theta}{2} = \text{RHS}
\end{aligned}$$