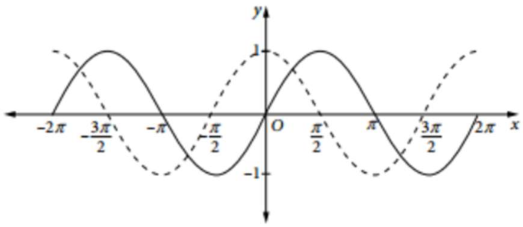
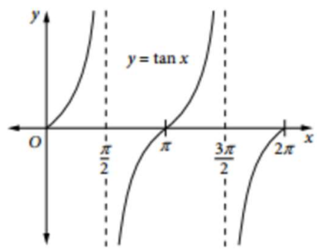
In the diagram below, the graph of $y = \sin x$ is the unbroken curve and the graph of $y = \cos x$ is the dashed curve. The graphs are drawn in the domain $-2\pi \le x \le 2\pi$. For a domain of real numbers, the graphs continue to repeat in both directions, so the graphs for the domains $-2\pi \le x \le 0$, $0 \le x \le 2\pi$ and $2\pi \le \theta \le 4\pi$ are the same.



As x increases, the values of $\sin x$ and $\cos x$ repeat themselves after an interval or **period** of 2π. Sine and cosine are therefore called **periodic functions**. This means that the points $P(\theta)$, $P(2\pi + \theta)$, $P(4\pi + \theta)$ on the unit circle all coincide and hence $\sin(2\pi + \theta) = \sin\theta$, $\cos(2\pi + \theta) = \cos\theta$ and so on.

The maximum and minimum values of $\sin x$ and $\cos x$ are 1 and -1 respectively, so we say that their **amplitude** is 1.

If the graph of $y = \cos x$ is translated $\frac{\pi}{2}$ units to the right, parallel with the x-axis, it coincides with the graph of $y = \sin\theta$. You can check that this follows from $\cos x = \sin\left(\frac{\pi}{2} + x\right)$. You can also check by sketching graphs by hand or by using graphing software.

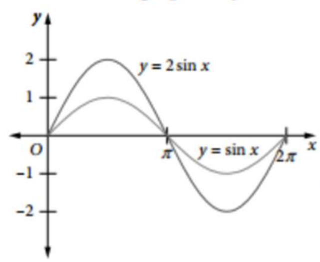


The diagram on the left shows the graph of $y = \tan x$ for $0 \le x \le 2\pi$. As x increases, the values of $\tan x$ repeat after an interval or **period** of π. Because $\tan x = \frac{\sin x}{\cos x}$, the graph of $y = \tan x$ does not exist when $\cos x = 0$, so the tan function is undefined at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

The domain of $\tan x$ is all real x except for $x = \frac{(2n-1)\pi}{2}$, where n is an integer (i.e. odd multiples of $\frac{\pi}{2}$). The range of $\tan x$ is all real y.

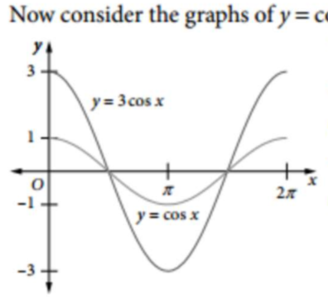
Graphs of $y = k\sin x$, $y = k\cos x$ and $y = k\tan x$

Consider the graphs of $y = \sin x$ and $y = 2\sin x$, drawn on the same axes for $0 \le x \le 2\pi$.



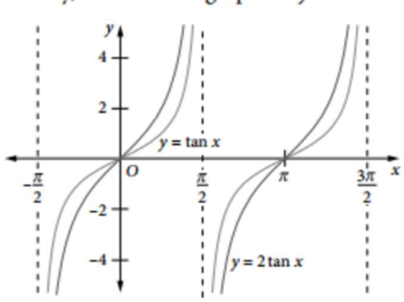
The graphs have the same shape and the same period, 2π. They both cut the x-axis at $x = 0, \pi, 2\pi$. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $2\sin x$ is 2, at $x = \frac{\pi}{2}$. The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $2\sin x$ is -2, at $x = \frac{3\pi}{2}$. The amplitude of $y = \sin x$ is 1 and the amplitude of $y = 2\sin x$ is 2.

Now consider the graphs of $y = \cos x$ and $y = 3\cos x$, drawn on the same axes for $0 \le x \le 2\pi$.



The graphs have the same shape and the same period, 2π. They both cut the x-axis at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $3\cos x$ is 3, at $x = 0, 2\pi$. The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $3\cos x$ is -3, at $x = \pi$. The amplitude of $y = \cos x$ is 1 and the amplitude of $y = 3\cos x$ is 3.

Finally, consider the graphs of $y = \tan x$ and $y = 2\tan x$, drawn on the same axes for $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.



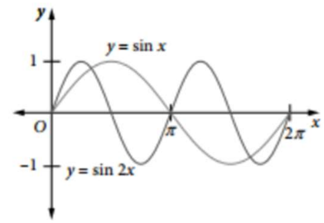
The graphs have the same shape and the same period, π. They both cut the x-axis at $x = 0, \pi$. The curves have no greatest or least value, so they have no amplitude. The curves are undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. The lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ are asymptotes.

The effect of the '2' in $y = 2\tan x$ is to make the y value for a particular value of x twice the corresponding y value for $\tan x$.

In general, the effect of the k multiplier on a trigonometric function is to multiply the function's amplitude by k. If the function has no amplitude, as for $y = k\tan x$, then the effect is to stretch the curve vertically by a factor of k.

Graphs of $y = \sin ax$, $y = \cos ax$ and $y = \tan ax$

Consider the graphs of $y = \sin x$ and $y = \sin 2x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

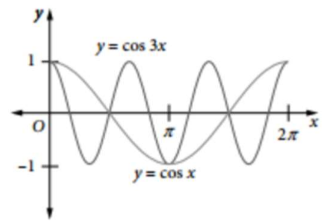


The graphs have the same shape, but in one period of $y = \sin x$ the graph of $y = \sin 2x$ occurs twice. This means that the period of $y = \sin 2x$ is half the period of $y = \sin x$. The period of $y = \sin 2x$ is π.

Both graphs cut the x-axis when $x = 0, \pi, 2\pi$. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $\sin 2x$ is also 1, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $\sin 2x$ is also -1, at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. The amplitude of both functions is 1.

Consider the graphs of $y = \cos x$ and $y = \cos 3x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

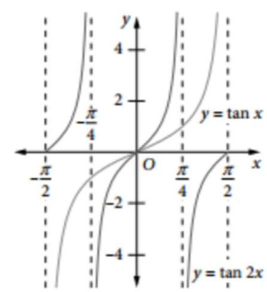


The graphs have the same shape, but in one period of $y = \cos x$ the graph of $y = \cos 3x$ occurs three times. This means that the period of $y = \cos 3x$ is one-third the period of $\cos x$. The period of $y = \cos 3x$ is $\frac{2\pi}{3}$.

Both graphs cut the x-axis at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $\cos 3x$ is also 1, at $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.

The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $\cos 3x$ is also -1, at $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$. The amplitude of both functions is 1.

Consider the graphs of $y = \tan x$ and $y = \tan 2x$, drawn on the same axes for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



The graphs have the same shape, but in one period of $y = \tan x$ the graph of $y = \tan 2x$ occurs twice. This means that the period of $y = \tan 2x$ is half the period of $\tan x$. The period of $y = \tan 2x$ is $\frac{\pi}{2}$.

Both graphs cut the x-axis when $x = 0$. The curves have no greatest or least value, so they have no amplitude. $y = \tan x$ is undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}$.

$y = \tan 2x$ is undefined at $x = -\frac{\pi}{4}, \frac{\pi}{4}$. The lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are asymptotes to $y = \tan x$, while the lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ are asymptotes to $y = \tan 2x$.

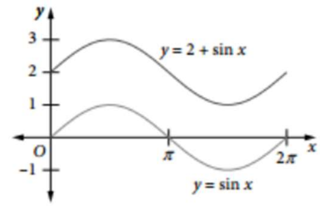
In general, the effect of the a multiplier in each equation is to divide the function's period by a. Thus the period of $y = \sin ax$ is $\frac{2\pi}{a}$, the period of $y = \cos ax$ is $\frac{2\pi}{a}$ and the period of $y = \tan ax$ is $\frac{\pi}{a}$.

Graphs of $y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$

Consider the graphs of $y = \sin x$ and $y = 2 + \sin x$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape and the same period, 2π. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $2 + \sin x$ is 3, at $x = \frac{\pi}{2}$. The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $2 + \sin x$ is 1, at $x = \frac{3\pi}{2}$.
The amplitude of both curves is 1.

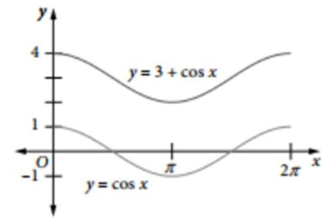
The effect of adding 2 to $\sin x$ is to translate the curve vertically upwards 2 units (i.e. 'move up 2').



Consider the graphs of $y = \cos x$ and $y = 3 + \cos x$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape and the same period, 2π. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $3 + \cos x$ is 4, at $x = 0, 2\pi$. The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $3 + \cos x$ is 2, at $x = \pi$.
The amplitude of both curves is 1.

The effect of adding 3 to $\cos x$ is to translate the curve vertically upwards 3 units (move up 3).



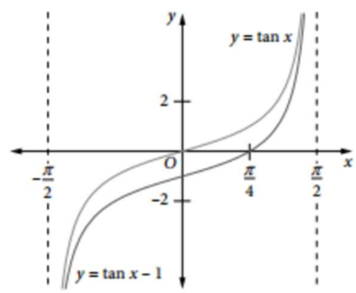
Consider the graphs of $y = \tan x$ and $y = \tan x - 1$, drawn on the same axes for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

The graphs have the same shape and the same period, π. The curves have no greatest or least value, so they have no amplitude.

The curves are undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}$. The lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are asymptotes.

The effect of the -1 in $y = \tan x - 1$ is to translate the curve vertically downwards 1 unit (move down 1).

In general, the effect of the c in each equation is to translate the curve vertically by c units.



Graphs of $y = k\sin a(x+b)$, $y = k\cos a(x+b)$ and $y = k\tan a(x+b)$

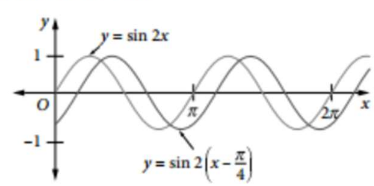
These graphs combine the effects seen so far. The value k is still the amplitude of the sine and cosine functions and affects the size of the y value in the tan function, and a still divides the period of the functions. The value b now changes the position along the x-axis where the functions occur. It is called the **phase** of the function.

Consider the graphs of $y = \sin 2x$ and $y = \sin 2\left(x - \frac{\pi}{4}\right)$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape, amplitude 1 and period π. The graph of $y = \sin 2x$ has been translated $\frac{\pi}{4}$ units to the right to obtain the graph of $y = \sin 2\left(x - \frac{\pi}{4}\right)$.

$y = \sin 2x$ cuts the x-axis when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

$y = \sin 2\left(x - \frac{\pi}{4}\right)$ cuts the x-axis at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The greatest value of each function is 1 and the least value of each function is -1.

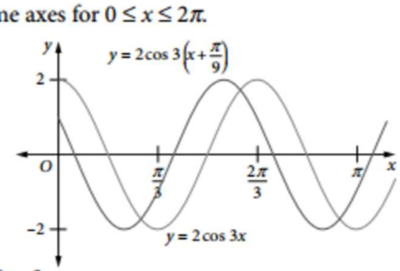


Consider the graphs of $y = 2\cos 3x$ and $y = 2\cos 3\left(x + \frac{\pi}{9}\right)$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape, amplitude 2 and period $\frac{2\pi}{3}$. The graph of $y = 2\cos 3x$ has been translated $\frac{\pi}{9}$ units to the left to obtain $y = 2\cos 3\left(x + \frac{\pi}{9}\right)$.

$y = 2\cos 3x$ cuts the x-axis when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

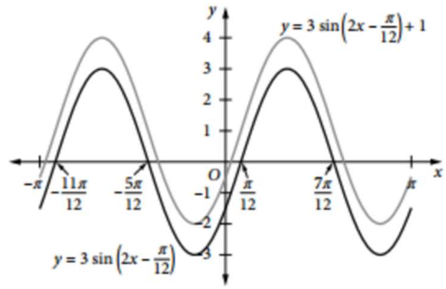
$y = 2\cos 3\left(x + \frac{\pi}{9}\right)$ cuts the x-axis when $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.



The greatest value of each function is 2 and the least value of each function is -2.

Graphs of $y = k\sin a(x+b)+c$, $y = k\cos a(x+b)+c$ and $y = k\tan a(x+b)+c$

These graphs have the same shape as $y = k\sin a(x+b)$, $y = k\cos a(x+b)$ and $y = k\tan a(x+b)$. The effect of the c is to change the y-value by c units, or graphically, to translate the graph vertically a distance of $|c|$ units, upwards if c is positive.



Consider the graphs of $y = 3\sin 2\left(x - \frac{\pi}{12}\right)$ and $y = 3\sin 2\left(x - \frac{\pi}{12}\right) + 1$ drawn on the same axes for $-\pi \le x \le \pi$.

The graphs have the same shape, amplitude 3 and period π. The graph of $y = 3\sin 2\left(x - \frac{\pi}{12}\right)$ has been translated one unit upwards to give the graph of $y = 3\sin 2\left(x - \frac{\pi}{12}\right) + 1$.

The curve $y = 3\sin 2\left(x - \frac{\pi}{12}\right)$ cuts the x-axis at $x = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$.

The curve $y = 3\sin 2\left(x - \frac{\pi}{12}\right) + 1$ cuts the line $y = 1$ at $x = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

The amplitude, period and phase of each function is the same. The greatest value of $3\sin\left(2x - \frac{\pi}{12}\right)$ is 3, its least value is -3. The greatest value of $y = 3\sin 2\left(x - \frac{\pi}{12}\right) + 1$ is 4, its least value is -2.

Summary of trigonometric functions

Function	Period	Amplitude	Domain	Range	Symmetry
$y = k\sin x$	2π	k	Real x	$-k \le y \le k$	Rotational symmetry
$y = k\cos x$	2π	k	Real x	$-k \le y \le k$	Symmetrical about y-axis
$y = k\tan x$	π	none	Real x, $x \ne \frac{(2n-1)\pi}{2}$ for integer n	Real y	Rotational symmetry
$y = k\sin ax$	$\frac{2\pi}{a}$	k	Real x	$-k \le y \le k$	Rotational symmetry
$y = k\cos ax$	$\frac{2\pi}{a}$	k	Real x	$-k \le y \le k$	Symmetrical about y-axis
$y = k\tan ax$	$\frac{\pi}{a}$	none	Real x, $x \ne \frac{(2n-1)\pi}{2a}$ for integer n	Real y	Rotational symmetry
$y = k\sin ax + c$	$\frac{2\pi}{a}$	k	Real x	$-k+c \le y \le k+c$	
$y = k\cos ax + c$	$\frac{2\pi}{a}$	k	Real x	$-k+c \le y \le k+c$	Symmetrical about y-axis
$y = k\tan ax + c$	$\frac{\pi}{a}$	none	Real x, $x \ne \frac{(2n-1)\pi}{2}$ for integer n	Real y	
$y = k\sin a(x+b)$	$\frac{2\pi}{a}$	k	Real x	$-k \le y \le k$	
$y = k\cos a(x+b)$	$\frac{2\pi}{a}$	k	Real x	$-k \le y \le k$	
$y = k\tan a(x+b)$	$\frac{\pi}{a}$	none	Real x, $x \ne \frac{(2n-1)\pi - 2c}{2a}$ for integer n	Real y	

When first sketching graphs of trigonometric functions, you should draw up a table of values and plot the points, joining the points with an appropriate curve. When you are familiar with the shapes of the standard curves, you will be able to draw them confidently using only the properties of amplitude, period and asymptotes. You might also use a standard template to obtain the shape of a curve and then mark the appropriate scale on the axes.

After you become more experienced at drawing these graphs by hand, draw them using graphing software and compare the results to check that you have not missed anything.

Example 1

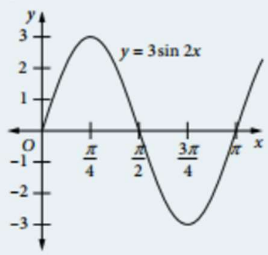
Sketch the graph of $y = 3\sin 2x$ for $0 \le x \le \pi$.

Solution

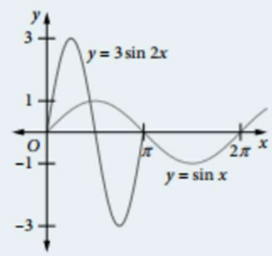
Complete a table of values.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
y	0	1.5	2.1	2.6	3	2.6	2.1	1.5	0	−1.5	−2.1	−2.6	−3	−2.6	−2.1	−1.5	0

Plot the points and sketch the graph, marking scales on each axis. Make sure that your curve is smooth, not pointed or jagged.



An alternative method is to first sketch $y = \sin x$ for $0 \le x \le 2\pi$. You can then draw $y = 3\sin 2x$ by noting that it has 3 times the amplitude and half the period of $y = \sin x$.



Example 2

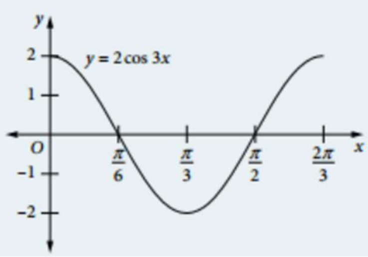
Sketch the graph of **(a)** $y = 2\cos 3x$ **(b)** $y = \tan \pi x$, showing one complete cycle.

Solution

(a) Amplitude = 2

Period $= \frac{2\pi}{3}$

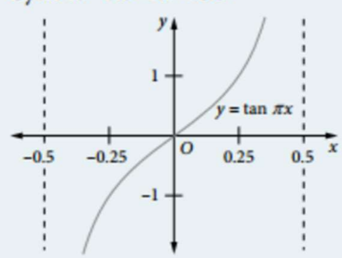
The midpoint of the cycle is $\frac{\pi}{3}$



(b) No amplitude

Period = 1 (variable is πx)

Cycle is $-0.5 < x < 0.5$



Example 3

(a) Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

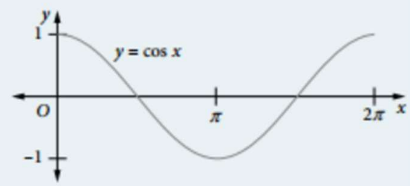
(b) On the same set of axes sketch $y = \cos\frac{x}{2}$.

(c) Hence sketch $y = -\cos\frac{x}{2}$.

(d) Use your answer to **(c)** to sketch $y = 1 - \cos\frac{x}{2}$.

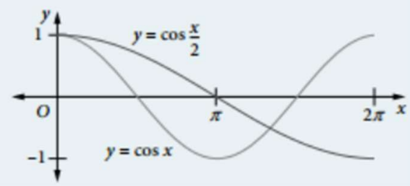
Solution

(a) $y = \cos x$ for $0 \le x \le 2\pi$.

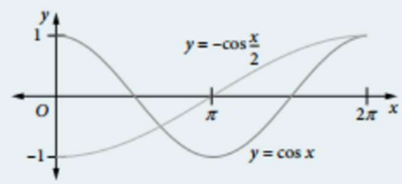


(b) The period of $y = \cos\frac{x}{2}$ is 4π, so half a cycle of the curve will fit in the domain $0 \le x \le 2\pi$.

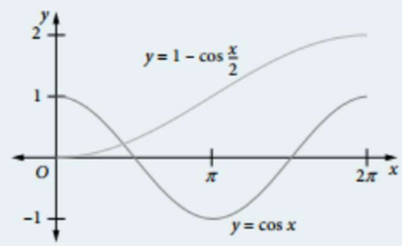
$y = \cos x$ and $y = \cos\frac{x}{2}$ both have an amplitude of 1.



(c) $y = -\cos\frac{x}{2}$ is just $y = \cos\frac{x}{2}$ flipped over (reflected in the x-axis).



(d) Obtain $y = 1 - \cos\frac{x}{2}$ by moving $y = -\cos\frac{x}{2}$ up 1 unit.



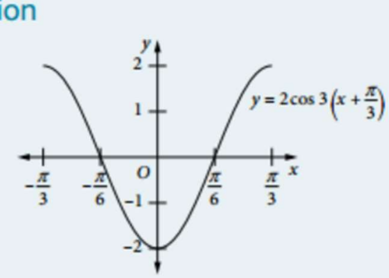
Example 4

Sketch graphs of the following, showing one complete cycle (period) of each:

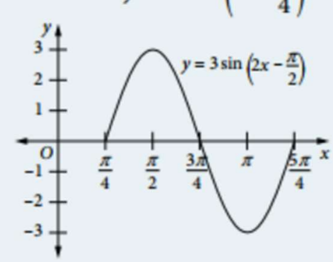
(a) $y = 2\cos 3\left(x + \frac{\pi}{3}\right)$ **(b)** $y = 3\sin\left(2x - \frac{\pi}{2}\right)$

Solution

(a)



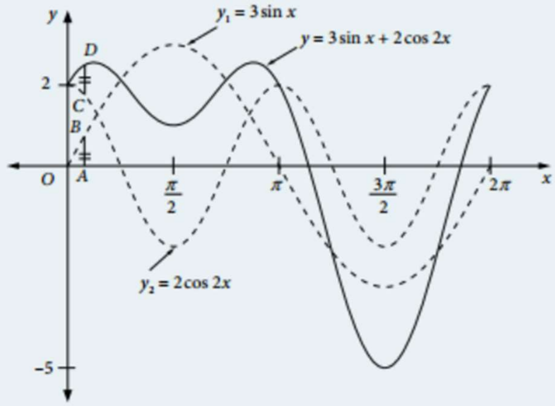
(b) Rewrite as $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$:



Example 5

Using the same scale and axes, sketch the graphs of $y = 3\sin x$ and $y = 2\cos 2x$. Hence sketch the graph of $y = 3\sin x + 2\cos 2x$ for $0 \le x \le 2\pi$.

Solution



The graphs of $y_1 = 3\sin x$ and $y_2 = 2\cos 2x$ are shown here as dashed lines. To obtain points on the graph of $y = 3\sin x + 2\cos 2x$ we can add the ordinates of the component curves, remembering to take the sign into account.

Let $y = y_1 + y_2$ where $y_1 = 3\sin x$ and $y_2 = 2\cos 2x$.

For example: at $x = 0$, $y = 0 + 2 = 2$; at $x = \frac{\pi}{2}$, $y = 3 - 2 = 1$; at $x = \pi$, $y = 0 + 2 = 2$; at $x = \frac{3\pi}{2}$, $y = -3 - 2 = -5$; at $x = 2\pi$, $y = 0 + 2 = 2$; and so on.

In general, at $x = A$, $AD = AB + AC$.

Example 6

Solve $3\sin\left(2x - \frac{\pi}{2}\right) = 1.5$ using graphical methods. Check your answer algebraically.

Solution

The solution is $x = \frac{\pi}{3}, \frac{2\pi}{3}$.

$$3\sin\left(2x - \frac{\pi}{2}\right) = 1.5$$

$$\sin\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}$$

$$2x - \frac{\pi}{2} = \frac{\pi}{6}, \pi - \frac{\pi}{6}$$

$$2x = \frac{\pi}{6} + \frac{\pi}{2}, \frac{5\pi}{6} + \frac{\pi}{2}$$

$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Note: Using graphing software, the scales can often be set to give the exact values in terms of π.