MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

1 Market research in a large city indicates that the maximum sales of a soon-to-be-released mobile device, is 10 truckloads per month (1 truckload = 10 000 devices).

Past experience with models iThingie1 through to iThingie6 indicates that the rate of growth in the truckloads of sales $\frac{ds}{dt}$, t months after the release of an iThingie, is directly proportional to the difference between the current sales and the maximum monthly sales.

(a) Find an equation for the rate of growth $\frac{ds}{dt}$ in the sales s as a function of the time t in months after the new product is first released onto the market. Express your answer in terms of the constant of proportionality r.

(b) Find the solution curve of your model. Express your answer in terms of the constant of proportionality r.

(c) If two truckloads are sold after one month, find the predicted number of truckloads per month after three months. (Express your answer correct to the nearest truckload.)

a) $\frac{ds}{dt} = r \cdot (10 - s)$

b) $\frac{ds}{10-s} = r\,dt \Rightarrow -\ln(10-s) = rt + C$

so $10 - s = e^{-rt+C}$

so $s = 10 - e^{-rt+C} = 10 - A e^{-rt}$

$s(0) = 0$ so $s(0) = 10 - A e^{-r \times 0}$ so $A = 10$

$s(t) = 10\left[1 - e^{-rt}\right]$

c) $s(1) = 2$ so as $s(1) = 10\left[1 - e^{-r}\right]$

we must have $10(1 - e^{-r}) = 2$

$1 - e^{-r} = \frac{2}{10} = \frac{1}{5}$ $e^{-r} = \frac{4}{5}$ $-r = \ln\left(\frac{4}{5}\right)$

So $s(t) = 10\left[1 - e^{t \ln \frac{4}{5}}\right]$

$s(t) = 10\left[1 - \left(\frac{4}{5}\right)^t\right]$

$s(3) = 10\left[1 - \left(\frac{4}{5}\right)^3\right] = 4.88$ about 5 truckloads.

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2 A simple model for the spread of a contagious illness assumes that the rate at which the illness spreads $\frac{dI}{dt}$ varies jointly with the product of the number of ill people I and the number of people still susceptible to the illness S. This means that $\frac{dI}{dt} = rIS$, $r > 0$.

Assume that one infected person is introduced into a fixed population of size P.
Then $P + 1 = I + S \therefore S = P + 1 - I$. Therefore, $\frac{dI}{dt} = rI(P+1-I)$, $I(0) = 1$ and $r > 0$.

(a) Show that $\frac{1}{I(P+1-I)} = \frac{1}{(1+P)}\left[\frac{1}{(1+P-I)} + \frac{1}{I}\right]$ **(b)** Find I as a function of time.

$\frac{dI}{dt} = rIS$ then $P+1 = I+S \therefore S = P+1-I$

$\therefore \frac{dI}{dt} = rI(P+1-I)$ and $I(0) = 1$

a) $\frac{dI}{I(P+1-I)} = r\,dt$ so $\int \frac{dI}{I(P+1-I)} = rt + C$ Ⓔ

We try splitting the term $\frac{1}{I(P+1-I)}$ as two fractions.

$$\frac{1}{I(P+1-I)} = \frac{a}{I} + \frac{b}{P+1-I} = \frac{a(P+1-I)+bI}{I(P+1-I)}$$

So we must have $b - a = 0$ and $a(P+1) = 1$ so $a = \frac{1}{P+1}$
$a = b$

$$\therefore \frac{1}{I(P+1-I)} = \frac{1}{p+1}\left[\frac{1}{I} + \frac{1}{P+1-I}\right]$$

b) Back to Ⓔ: $\Leftrightarrow \frac{1}{P+1}\left[\int \frac{dI}{I} + \int \frac{dI}{P+1-I}\right] = rt + C$

$\Leftrightarrow \frac{1}{P+1}\left[\ln I - \ln(P+1-I)\right] = rt + C$

$\Leftrightarrow \ln\left[\frac{I}{P+1-I}\right] = r(P+1)t + D$ so $\frac{I}{P+1-I} = Ae^{r(P+1)t}$

$I = (P+1-I)Ae^{r(P+1)t}$ so $I\left[1 + Ae^{r(P+1)t}\right] = A(P+1)e^{r(P+1)t}$

$I = \frac{A(P+1)e^{r(P+1)t}}{1+Ae^{r(P+1)t}} = \frac{P+1}{1+Be^{-r(P+1)t}}$

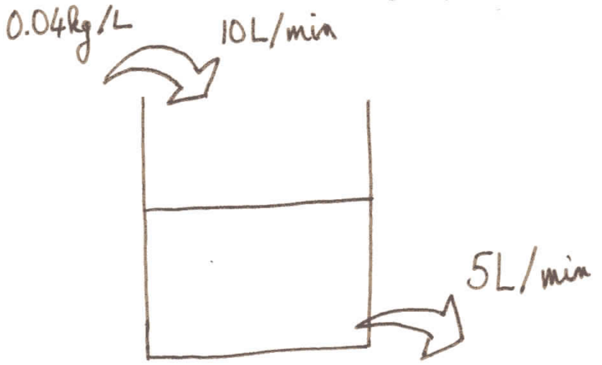
But $I(0) = 1$
so $I(0) = \frac{1+P}{1+B}$ so $B = P$

$I = \frac{(1+P)}{1+Pe^{-r(P+1)t}}$

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

4 A tank initially contains 1000 litres of salt solution of concentration 0.01 kg/L. A solution of the same salt, but concentration 0.04 kg/L, flows into the tank at a rate of 10 litres per minute. The mixture in the tank is kept uniform by stirring and the mixture flows out at a rate of 5 litres per minute.

Let Q kg be the quantity of salt in the tank after t minutes. Set up (but do not solve) the differential equation for Q in terms of t, and specify the initial conditions.



The initial quantity (mass) of salt in the tank is 0.01 kg/L × 1000 L = 10 kg.

More water is coming inside the tank than what's leaving, so

$V(t) = 1000 + 10 \times t - 5t = 1000 + 5t$ at any time t.

Let $Q(t)$ be the quantity of salt in the tank at any time t.

$$\frac{dQ}{dt} = \text{Quantity of salt that comes in} - \text{Quantity of salt that comes out}$$

$$\frac{dQ}{dt} = 0.04 \times 10 - \left(\frac{Q(t)}{V(t)}\right) \times 5$$

this quantity is the concentration of salt in the outflow

$$\boxed{\therefore \frac{dQ}{dt} = 0.4 - \frac{5Q}{1000 + 5t}}$$

and we also know that at $t = 0$, $Q(0) = 10$ kg.

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

5 Carbon monoxide (chemical symbol CO) is toxic to humans. Two hours of exposure to air with a volume concentration of CO at 0.02% will cause headaches and confusion.

During World War I, some generals commanded their soldiers from inside a bombproof bunker with an internal volume of 80 m^3. Troops resting near the air intake to the bunker would often smoke cigarettes. Unfortunately, the air intake to the bunker sucked the carbon-monoxide-filled smoke from the cigarettes back into the bunker.

Assume that smoky air is sucked into the bunker at a rate of 2 m^3/min, and that 0.03% of this air (by volume) is carbon monoxide. Ventilation fans keep the air well mixed inside the bunker, and the well-mixed air is extracted from the bunker at the same rate of 2 m^3/min. It can be shown that $\frac{dv}{dt} = 0.025(0.0003 - v)$, $v(0) = 0$.

(a) Solve this differential equation to find $v(t)$.

(b) Hence find the time for the volume fraction of carbon monoxide to reach 0.02% by volume inside the bunker. Express your answer in minutes, correct to the nearest minute.

a) $\frac{dV}{dt} = 0.025\,(0.0003 - V) \Rightarrow \frac{dv}{0.0003 - v} = 0.025\,dt$

$\Rightarrow -\ln(0.0003 - v) = 0.025\,t + C$

$\Rightarrow 0.0003 - v = A\,e^{-0.025t}$

$\Rightarrow V = 0.0003 - A\,e^{-0.025t}$

But $v(0) = 0$, so $v(0) = 0.0003 - A$ so $A = 0.0003$

$V(t) = 0.0003\left[1 - e^{-0.025t}\right]$

b) For v to be 0.0002, we need to solve:

$0.0002 = 0.0003\left[1 - e^{-0.025t}\right]$

$\Leftrightarrow 1 - e^{-0.025t} = \frac{2}{3} \Leftrightarrow e^{-0.025t} = \frac{1}{3}$

$\Leftrightarrow -0.025t = \ln(1/3) \Rightarrow t = \frac{\ln 3}{0.025} \simeq 44$ minutes

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

6 A lottery winner puts \$5 000 000 in winnings into a fund that has a 5% annual rate of return, paid continuously throughout the year. Each year the winner spends \$300 000, withdrawn from the account at a continuous rate over the course of the year.

(a) Show that the differential equation to model the fund balance $x(t)$ after t years is $\frac{dx}{dt} = 0.05(x - 6000000)$ and state the value of $x(0)$.

(b) Solve the differential equation in part **(a)**.

(c) Hence determine the balance after 20 years. Express your answer correct to the nearest 5 cents.

a) let x be the money in the fund.

Variation of money in the fund = income − outcome

$$\frac{dx}{dt} = x \times 0.05 - 300,000 = 0.05(x - 6,000,000)$$

with $x(0) = 5,000,000$.

b) $$\frac{dx}{x - 6\times10^6} = 0.05dt \quad \text{so} \int \frac{dx}{x-6\times10^6} = 0.05t + C$$

$$\Rightarrow \ln(6\times10^6 - x) = 0.05t + C$$

$$\Rightarrow 6\times10^6 - x = Ae^{0.05t}$$

$$\Rightarrow x = 6\times10^6 - Ae^{0.05t}$$

at $t=0$, $x = 5\times10^6$ so $A = 10^6$

$$x(t) = 10^6(6 - e^{0.05t})$$

c) $$x(20) = 10^6(6 - e^{0.05\times20}) = 3,281,718$$

Balance after 20 years.

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

7 Which of the following is not a model for exponential growth y?

A $y=\frac{10}{e^{-t}}, t>0$ $\quad y = 10e^{t}$

[✓ **B**] $y=10(0.5)^{t}, t>0$ ← Not growth because $0.5<1$

C $\log_e \frac{1}{y}=\log_e 10-2t, t>0$

$$\frac{1}{y} = \exp(\ln 10 - 2t) = 10e^{-2t}$$

$$y = \frac{1}{10}e^{2t}$$

D $\log_e(y)=\log_e(10)+t, t>0$

$$y = \exp(\ln 10 + t) = e^{\ln 10 + t} = 10e^{t}$$

9 The rate at which a rumour spreads throughout a population of 1000 students is proportional to the product of the number N of students who know the rumour and the number of students who haven't yet heard the rumour after t hours. If two students decide to start a rumour, the model that best decribes the spread of the rumour t hours later is:

A $\frac{dN}{dt}=k\frac{(1000-N)}{1000}, \quad N(0)=0$

B $\frac{dN}{dt}=k(N-2)(1000-N), \quad N(0)=2$

C $\frac{dN}{dt}=kN(1000-N), \quad N(0)=0$

[✓ **D**] $\frac{dN}{dt}=kN(1000-N), \quad N(0)=2$ ← 2 students start the rumour.

10 A chemical dissolves in a pool at a rate equal to 10% of the amount of undissolved chemical. Initially the amount of undissolved chemical is 5 kg and after t hours x kilograms has dissolved. The differential equation that models this process is:

A $\frac{dx}{dt}=\frac{x}{10}$ $\quad$ [✓ **B**] $\frac{dx}{dt}=\frac{5-x}{10}$ $\quad$ **C** $\frac{dx}{dt}=\frac{x-5}{10}$ $\quad$ **D** $\frac{dx}{dt}=5-\frac{x}{10}$

11 The rate of increase in the number of bacteria in a laboratory is directly proportional to the number present. If the number of bacteria triples every 2 hours, after how many hours will the number of bacteria be quadruple its initial value?

A $2\log_e\frac{4}{3}$ $\quad$ [✓ **B**] $\frac{2\log_e 4}{\log_e 3}$ $\quad$ **C** $\left(\frac{\log_e 3}{2}\right)^2$ $\quad$ **D** $\frac{\log_e 3}{\log_e 2}$

$\frac{dN}{dt} = kN$ so $\frac{dN}{N} = k\,dt$ $\quad \ln N = kt + c$ $\quad \boxed{N = Ae^{kt}}$

At $t=0$, $N(0)=A$ so $N = N(0)e^{kt}$

At $t=2$, $N(2) = 3N(0) = N_0 e^{2k}$ so $e^{2k} = 3$ $\quad 2k = \ln 3$, $k = \frac{\ln 3}{2}$

$N = N(0)e^{\frac{\ln 3 t}{2}} = N(0)3^{t/2}$

when $N = 4N(0)$: $\quad 4 = 3^{t/2}$ $\quad \ln 4 = \frac{t}{2}\ln 3$ $\quad t = 2\left(\frac{\ln 4}{\ln 3}\right)$

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

15 At any time $t \geq 0$ (in days), the rate of growth in the number of bacteria in a laboratory is directly proportional to the number N currently present. The initial population of bacteria is 1000.

(a) Assuming this growth rate continues indefinitely, write a differential equation to model the number of bacteria present in the dish after $t \geq 0$ days.

The initial population of 1000 bacteria triples during the first 2 days.

(b) Hence, show that $N(t) = a \times 3^{\frac{t}{b}}$ for a suitable choice of the positive integers a and b.

(c) By what factor will the population have increased in the first 4 days?

(d) How much time will it take for the population to grow to 10 times its initial value? Express your answer correct to the nearest hour.

a) $N(0) = 1{,}000$ $\quad \frac{dN}{dt} = kN \quad \Rightarrow \frac{dN}{N} = k\,dt$

$\Rightarrow \ln N = kt + C \quad \Rightarrow N = Ae^{kt}$

But $N(0) = A = 1{,}000$ so $N = 1000\,e^{kt}$

At $t = 2$ $\quad N = 3{,}000$ so $3000 = 1000\,e^{2k}$

so $e^{2k} = 3 \quad \Rightarrow 2k = \ln 3 \quad \Rightarrow k = \ln\sqrt{3}$

So $N = 1000\,e^{t \times \ln\sqrt{3}} = 1000\,e^{\ln\sqrt{3}}e^{t} = 1000\sqrt{3}\,e^{t}$

or $N = 1000\left(\sqrt{3}\right)^{t}$ as $a^{x} = e^{x\ln a}$

or $N = 1000 \times 3^{t/2}$

c) At $t = 4$ $\quad N(4) = 1000 \times 3^{4/2} = 1000 \times 3^{2} = 9 \times N(0)$

so factor 9.

d) $N(t) = 10{,}000$ when $1000 \times 3^{t/2} = 10{,}000$

$\Rightarrow 3^{t/2} = 10 \quad \Rightarrow \ln 3^{t/2} = \ln 10 \Rightarrow \frac{t}{2}\ln 3 = \ln 10$

so $t = \frac{2\ln 10}{\ln 3} = \frac{\ln 100}{\ln 3} \sim 4.19..$

4 days 5 hours.

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

16 Almost all carbon in the world is carbon-12, which is the most common stable 'isotope' (nuclear form) of carbon. In the late 1940s the American scientist Willard Libby studied carbon-14, which is not stable: it radioactively decays according to the reaction $^{14}C \to {}^{14}N + e^- + \bar{\nu}_e$, in which a neutron spontaneously transforms into a proton (thus changing the atom from carbon C to nitrogen N) as it emits an electron and an antineutrino. In the upper atmosphere, carbon-12 sometimes transforms back into carbon-14 due to interactions with cosmic rays, so the proportion of both isotopes in the atmosphere stays relatively constant.

But whenever carbon is absorbed by plants to become part of living organisms in the world the carbon-12 is mostly shielded from transforming into carbon-14. This means that when an organism dies, its concentration of carbon-12 $\left(\left[{}^{12}C\right]\right)$ remains relatively constant, but its concentration of carbon-14 $\left(\left[{}^{14}C\right]\right)$ radioactively decays at the rate $\frac{d\left[{}^{14}C\right]}{dt} = -r\left[{}^{14}C\right]$, $r = 1.2097 \times 10^{-4}$ years^{-1}.

(a) Find the half-life $\left(t_{\frac{1}{2}}\right)$ of carbon-14, correct to the nearest year.

The ratio of carbon-12 to carbon-14 remains relatively constant in living organisms, roughly $R = \frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]} \approx 1.3 \times 10^{-12}$, but this ratio changes after the organism dies (because it stops absorbing new carbon-14 atoms from the atmosphere, while any carbon-14 present is still decaying). Consequently, you can determine the length of time since an organism's death by measuring how much this ratio has changed.

(b) Half the original carbon-14 has radioactively decayed. How many years ago did the tree die?

(c) Find a differential equation for the ratio $R = \frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]}$. (Hint: $\left[{}^{12}C\right]$ can be considered a constant.)

(d) Hence find a formula for the ratio R after t years in the form $R(t) = a\left(\frac{1}{2}\right)^{\frac{t}{n}}$ for a suitable choice of real number a and integer n.

(e) The skeleton of an extinct mega-marsupial is found to have a carbon ratio $\frac{\left[{}^{14}C\right]}{\left[{}^{12}C\right]} = 0.9 \times 10^{-4}$. How many years ago, to the nearest year, did the animal die?

a) Note $[{}^{14}C] = C$ $\quad \frac{dC}{dt} = -rC \Rightarrow \frac{dC}{C} = -r\,dt$

So $\ln C = -rt + D \Rightarrow C = Ae^{-rt} = C_0 e^{-rt}$

At half life $t_{1/2}$, $C = \frac{C_0}{2}$ $\quad C(t_{1/2}) = C_0 e^{-rt_{1/2}}$

So $\frac{1}{2} = e^{-rt_{1/2}} \Rightarrow \ln\frac{1}{2} = -rt_{1/2} \Rightarrow t_{1/2} = \frac{\ln 2}{r}$

So $t_{1/2} = \frac{\ln 2}{1.2097 \times 10^{-4}} = 5730$ years.

b) if half of the original carbon 14 has disappeared, that means we are at half life, i.e. 5730 years.

c) $R = \frac{[^{14}C]}{[^{12}C]}$ $\frac{dR}{dt} = \frac{d}{dt}\left[\frac{[^{14}C]}{[^{12}C]}\right] = \frac{1}{[^{12}C]}\frac{d}{dt}[^{14}C]$

so $\frac{dR}{dt} = \frac{1}{[^{12}C]} \times \left(-r[^{14}C]\right) = -rR. \Rightarrow \frac{dR}{R} = -r\,dt$

d) $\ln R = -rt + D \Rightarrow R = Ae^{-rt}$

At t_0, $R = 1.3 \times 10^{-12}$ so $R = 1.3 \times 10^{-12} e^{-rt}$

Further $r = \frac{-\ln 2}{t_{1/2}}$ $\therefore$ $R = 1.3 \times 10^{-12} e^{-\ln 2\, t/t_{1/2}}$

$R = 1.3 \times 10^{-12} \times 2^{-(t/t_{1/2})} = 1.3 \times 10^{-12} \times \left(\frac{1}{2}\right)^{(t/5370)}$

$a = 1.3 \times 10^{-12}$ and $n = 5370$

e) Here $R = 0.9 \times 10^{-4}$

$\frac{R}{1.3 \times 10^{-12}} = \left(\frac{1}{2}\right)^{t/5370} \Rightarrow \ln\left(\frac{R}{1.3 \times 10^{-12}}\right) = \frac{t}{5370} \ln\left(\frac{1}{2}\right)$

$\Rightarrow t = 5370 \times \left[\frac{\ln(R/1.3 \times 10^{-12})}{-\ln 2}\right]$

So $t = 5370 \times \left[\frac{\ln(0.9 \times 10^{-4}/1.3 \times 10^{-12})}{-\ln 2}\right]$

$t = -139{,}861$ years ago.

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

17 Bubonic plague (known as the Black Death) ravaged Europe most severely between the years 1347 and 1351. It is estimated that the plague killed more than one in three people living in Europe at that time. The disease was survivable only in about 5% of cases.

Epidemiologists commonly use S to represent the fraction of a population that have survived an epidemic disease, t days after its arrival. The change in this fraction over time can be modelled by the differential equation:

$\frac{dS}{dt} = -r(S-I)$, $S(0)=1$, where I is the fraction of the population that ultimately recover (and hence survive).

(a) Find a formula for the survivability fraction S as a function of the time t days.

(b) Show that $\frac{d^2S}{dt^2} = r^2(S-I)$

After 1 month, the survivability fraction S approaches a steady state value of 0.05.

(c) Find I.

(d) Find the value of r if after 14 days, only 6% of the population has survived.

(e) Find the time when the death rate reaches its maximum value and state this death rate.

(f) Plot the survivability fraction over the first 2 weeks.

a) $\frac{dS}{dt} = -r(S-I) \Rightarrow \frac{dS}{S-I} = -r\,dt$

$\ln|S-I| = -rt + C$ so $S-I = Ae^{-rt}$

$S = I + Ae^{-rt}$ $S(0)=1$ so $1 = I + A$ so $A = 1-I$

$S = I + (1-I)e^{-rt}$

b) $\frac{dS}{dt} = -rS + rI$ so $\frac{d^2S}{dt^2} = -r\frac{dS}{dt} = -r \times (-r(S-I))$

$\frac{d^2S}{dt^2} = r^2(S-I)$

c) S approaches a steady value of 0.05, so after one month $\frac{dS}{dt} = 0$

i.e., $-r(S-I) = 0$ or $S = I$ so $I = 0.05$

d) After 14 days, $S = 0.06$

$S(t) = I + (1-I)e^{-rt} \Rightarrow \frac{S-I}{1-I} = e^{-rt}$

so $-rt = \ln\left(\frac{S-I}{1-I}\right)$ so $r = \frac{1}{t}\ln\left(\frac{1-I}{S-I}\right)$

$$r = \frac{1}{14} \ln\left(\frac{1-0.05}{0.06-0.05}\right) = \frac{1}{14} \ln\left(\frac{0.95}{0.01}\right) = \frac{1}{14} \ln 95 \simeq 0.3253$$

e) The death rate reaches its maximum value when $\frac{dS}{dt} = 0$

so when $S = I$ as $\frac{dS}{dt} = -r(S-I)$

when $S = I$, as $S(t) = I + (1-I)e^{-rt}$

$$\Rightarrow \underline{I} = I + (1-I)e^{-rt}$$

$$\Rightarrow (1-I)e^{-rt} = 0$$

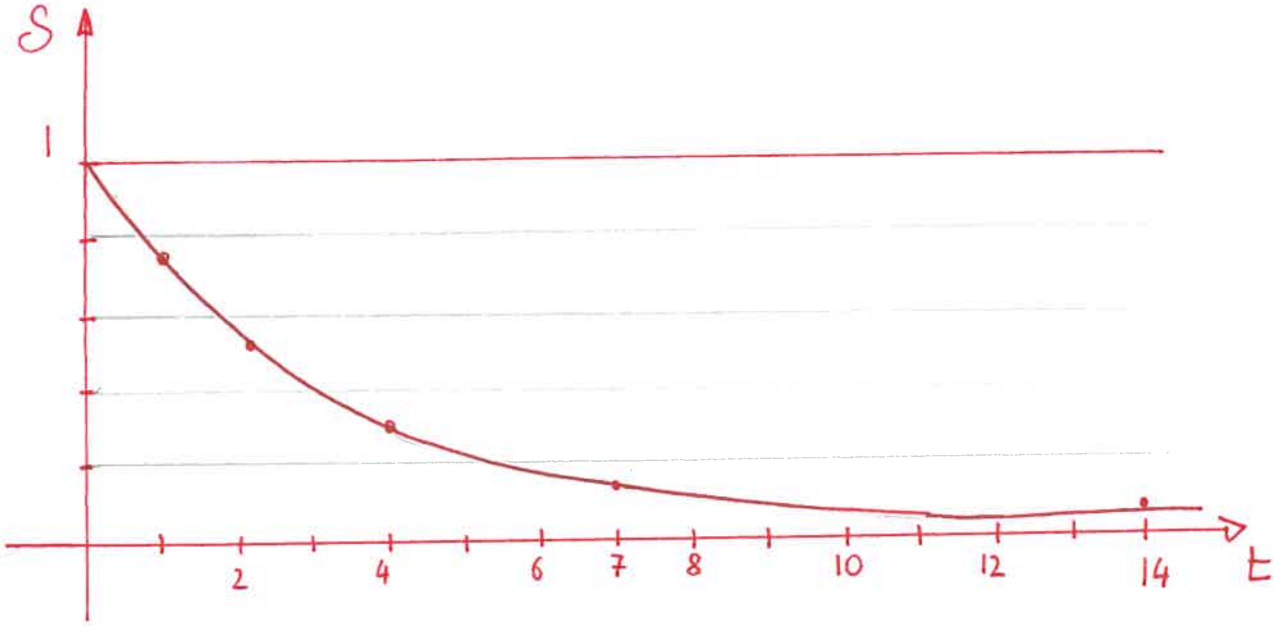
so $I = 1$ as $e^{-rt} \neq 0$

So the maximum death rate is $S = 1$

This occurs when $1 = I + (1-I)e^{-rt} \Rightarrow e^{-rt} = 1$

so when $-rt = 0$ or $t = 0$ (at the begining of the pandemic)

f) $S = 0.05 + 0.95\, e^{-0.3253\, t}$



18 Two types of bacteria, type A and type B, coexist in a biological system. Assume that each population grows exponentially. The proportion of the total bacteria population belonging to type A is given by $p(t)=\dfrac{A(t)}{A(t)+B(t)}$. ①

(a) Express the growth rate $\frac{dp}{dt}$ in terms of $A(t)$, $B(t)$, $A'(t)$ and $B'(t)$.

(b) Given that $A'(t)=r_A A(t)$, $r_A>0$ and $B'(t)=r_B B(t)$, $r_B>0$, express the growth rate $\frac{dp}{dt}$ in terms r_A, r_B, A and B.

(c) Hence write the growth rate $\frac{dp}{dt}$ in terms r_A, r_B and p.

(d) If $p(0)=\frac{1}{10}$ and $r_A-r_B=\frac{1}{100}$ hour^{-1}, and given that $\frac{1}{p(1-p)}=\frac{1}{p}+\frac{1}{1-p}$, find $p(10)$.

a) $$\frac{dp}{dt}=\frac{A'(t)\left[A(t)+B(t)\right]-\left[A'(t)+B'(t)\right]A(t)}{\left[A(t)+B(t)\right]^2}=\frac{A'(t)\,B(t)-B'(t)\,A(t)}{\left[A(t)+B(t)\right]^2}$$

b) $A'(t)=r_A\,A(t)$ and $B'(t)=r_B\,B(t)$

So $$\frac{dp}{dt}=\frac{r_A\,A(t)B(t)-r_B\,B(t)A(t)}{\left[\frac{A(t)}{p(t)}\right]^2}=\left(p(t)\right)^2\left[\frac{r_A\,B(t)-r_B\,B(t)}{A(t)}\right]$$

So $$\frac{dp}{dt}=p^2\left(r_A-r_B\right)\frac{B(t)}{A(t)}$$

c) From ① $p(t)=\dfrac{1}{1+B(t)/A(t)}$ so $1+\dfrac{B(t)}{A(t)}=\dfrac{1}{p(t)}$

So $$\frac{B(t)}{A(t)}=\frac{1}{p(t)}-1=\frac{1-p(t)}{p(t)}$$

So $$\frac{dp}{dt}=p(1-p)(r_A-r_B)\Rightarrow\frac{dp}{p(1-p)}=(r_A-r_B)\,dt$$

d) $$\int\frac{dp}{p(1-p)}=(r_A-r_B)t+C\Rightarrow\int\frac{dp}{p}+\int\frac{dp}{1-p}=(r_A-r_B)t+C$$

$$\Rightarrow\ln\left(\frac{p}{1-p}\right)=(r_A-r_B)t+C$$

At $t=0$ $p(0)=1/10$ so $\ln\left(\frac{0.1}{1-0.1}\right)=\frac{1}{100}\times 0+C$ so $C=\ln\left(\frac{1}{9}\right)=-\ln 9$

So $$\ln\left(\frac{p}{1-p}\right)=(r_A-r_B)t-\ln 9$$

when $t=10$ $\ln\left(\frac{p}{1-p}\right)=\frac{1}{100}\times 10-\ln 9=0.1-\ln 9$

which gives $\frac{p}{1-p}=0.123$ $p(1+0.123)=0.123$ $p=0.10937$