HARDER EXPONENTIAL GROWTH AND DECAY

1 N is decreasing according to the equation $\frac{dN}{dt} = -0.4(N-30)$. If $N = 60$ when $t = 0$:

(a) show that $N = 30 + Ae^{-0.4t}$ is a solution of this equation, where A is a constant

(b) calculate the value of N when $t = 5$.

a) $N = 30 + Ae^{-0.4t}$ so $\frac{dN}{dt} = A \times (-0.4)\, e^{-0.4t}$

whereas: $-0.4(N-30) = -0.4\left[(30 + Ae^{-0.4t}) - 30\right]$

$= -0.4\left[\cancel{30} + Ae^{-0.4t} - \cancel{30}\right]$

$= -0.4 \times A\, e^{-0.4t}$

$= \frac{dN}{dt}$

So indeed, $N = 30 + Ae^{-0.4t}$ is a solution of this equation.

b) when $t = 0$, $N = 60$, so $60 = 30 + Ae^{-0.4 \times 0}$

$\Leftrightarrow 60 = 30 + A$ so $A = 30$

So $N(t) = 30 + 30e^{-0.4t}$

At $t = 5$ $N(5) = 30 + 30e^{-0.4 \times 5}$

$N(5) = 30 + 30e^{-2}$

$N(5) = 30\left(1 + \frac{1}{e^2}\right)$

$N(5) \simeq 34.06$

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3 The original temperature of a body is 120°C, the temperature of its surroundings is 50°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-50)$ where T is the temperature of the body at time t, find:

(a) the temperature after 20 minutes **(b)** the time taken to cool to 60°C.

a) $\frac{dT}{dt} = -k(T-50)$ so the general solution is $T(t) = 50 + Ae^{-kt}$

At $t = 0$, $T = 120°C$ so $120 = 50 + A$ so $A = 70$
$T = 50 + 70e^{-kt}$

At $t = 10$, $T = 70°C$ so $70 = 50 + 70e^{-k\times 10} \Leftrightarrow e^{-10k} = \frac{20}{70} = \frac{2}{7}$

so $-10k = \ln\frac{2}{7}$ so $k = \frac{\ln(7/2)}{10}$

$\therefore T(t) = 50 + 70e^{\ln(2/7)\times\frac{t}{10}} = 50 + 70\left(\frac{2}{7}\right)^{t/10}$ $T(t) = 50 + 70\left(\frac{2}{7}\right)^{t/10}$

At $t = 20$ $T(20) = 50 + 70\left(\frac{2}{7}\right)^{20/10} \simeq 55.7°C$

b) For $T = 60°C$ $60 = 50 + 70\left(\frac{2}{7}\right)^{t/10} \Leftrightarrow \left(\frac{2}{7}\right)^{t/10} = \frac{10}{70} = \frac{1}{7}$

so $\ln\left[\left(\frac{2}{7}\right)^{t/10}\right] = \ln\left(\frac{1}{7}\right) \Leftrightarrow \frac{t}{10}\ln\left(\frac{2}{7}\right) = \ln\left(\frac{1}{7}\right)$

$\Leftrightarrow t = 10\left[\frac{\ln 1 - \ln 7}{\ln 2 - \ln 7}\right] = 10\left[\frac{\ln 7}{\ln 7 - \ln 2}\right] \simeq 15.53$ or $15'32''$

4 If $N = 70$ when $t = 0$, which expression is the correct solution to $\frac{dN}{dt} = -0.5(N-20)$?

A $N = 20 + 50e^{0.5t}$ $\boxed{\textbf{B}}$ $N = 20 + 50e^{-0.5t}$ **C** $N = 20 - 50e^{0.5t}$ **D** $N = 20 - 50e^{-0.5t}$

The general solution is $N(t) = 20 + Ae^{-0.5t}$

At $t = 0$, $N(0) = 70 = 20 + Ae^{0} = 20 + A$ so $A = 50$

$N(t) = 20 + 50e^{-0.5t}$ $\boxed{B}$

5 A metal bar has a temperature of 1230°C and cools to 1030°C in 10 minutes when the surrounding temperature is 30°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-30)$ where T is the temperature of the body at time t.

(a) Show that $T = 30 + 1200e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.

(b) Find the temperature after 20 minutes. **(c)** Find the time taken to cool from 1230°C to 80°C.

a) $T = 30 + 1200e^{-kt}$ so $\frac{dT}{dt} = 1200 \times (-k) e^{-kt} = -1200k\, e^{-kt}$

whereas $-k(T-30) = -k(\cancel{30} + 1200\, e^{-kt} - \cancel{30}) = -1200k\, e^{-kt}$

So indeed, $T = 30 + 1200\, e^{-kt}$ satisfies Newton's law of cooling

Also, at $t=0$, $T(0) = 30 + 1200\, e^{-k\times 0} = 1230$ which is correct

So it satisfies both Newton's law of cooling and the initial conditions.

b) At $t=10$, $T(10) = 1030 = 30 + 1200\, e^{-k\times 10}$

so $e^{-k\times 10} = \frac{1000}{1200} = \frac{5}{6}$ so $k = \frac{-1}{10}\ln\left(\frac{5}{6}\right) = \frac{1}{10}\ln\left(\frac{6}{5}\right)$

so $T = 30 + 1200\, e^{\ln(5/6)\times t/10} = 30 + 1200 \times \left(\frac{5}{6}\right)^{t/10}$

At $t = 20$, $T(20) = 30 + 1200\left(\frac{5}{6}\right)^{20/10} = 863°C$

c) For $T = 80$, we must have $80 = 30 + 1200\left(\frac{5}{6}\right)^{t/10}$

$\Leftrightarrow \left(\frac{5}{6}\right)^{t/10} = \frac{50}{1200} = \frac{1}{24}$ so $\ln\left(\frac{5}{6}\right)^{t/20} = \ln\left(1/24\right)$

$\Leftrightarrow \frac{t}{10} = \frac{-\ln 24}{\ln 5 - \ln 6}$ so $t = 10\left[\frac{\ln 24}{\ln 6 - \ln 5}\right]$

$t = 174$ minutes or 2 hours 54'

7 A body whose temperature is 180°C is immersed in a liquid that is at 60°C. In 1 minute the temperature of the body has fallen to 120°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-60)$ where T is the temperature of the body at time t.

(a) Show that $T = 60 + 120e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.

(b) At what time would the temperature of the body have fallen to 90°C?

a) $T = 60 + 120\, e^{-kt}$

At $t=0$, $T = 60 + 120 \times e^0 = 180$ so it satisfies the initial conditions.

$\frac{dT}{dt} = 120 \times (-k)\, e^{-kt}$

whereas: $-k(T-60) = -k\left[\cancel{60} + 120\, e^{-kt} - \cancel{60}\right] = -120 k\, e^{-kt}$

So indeed $\frac{dT}{dt} = -k(T-60)$ when $T = 60 + 120\, e^{-kt}$

So $T = 60 + 120\, e^{-kt}$ satisfies both the initial conditions and Newton's law of cooling

b) For $T = 90$, we must have: $90 = 60 + 120\, e^{-kt}$

At $t=1$, $T = 120$ so $120 = 60 + 120\, e^{-k} \Leftrightarrow e^{-k} = \frac{60}{120} = \frac{1}{2}$

so $-k = \ln(1/2)$ or $k = \ln 2$

$T(t) = 60 + 120\, e^{-\ln 2 \times t} = 60 + 120 \times 2^{-t}$

So $90 = 60 + 120 \times 2^{-t} \Rightarrow 2^{-t} = \frac{30}{120} = \frac{1}{4}$

So $-t \times \ln 2 = \ln(1/4) = -\ln 4 = -\ln 2^2 = -2 \ln 2$

So $t = 2$ minutes.

8 A current of i amperes (or 'amps') flows through a coil of inductance L henrys and resistance R ohms. The current at any time is given by $i=\frac{E}{R}\left(1-e^{\frac{-Rt}{L}}\right)$, where E is the electromotive force (i.e. the voltage) in volts. Show that $L\frac{di}{dt}+Ri=E$.

$$i=\frac{E}{R}\left(1-e^{-Rt/L}\right)=\frac{E}{R}-\frac{E}{R}e^{-\frac{R}{L}t}$$

$$\text{so } \frac{di}{dt}=-\frac{E}{R}\times\left(-\frac{R}{L}\right)e^{-\frac{R}{L}t}=\frac{E}{L}e^{-\frac{R}{L}t}$$

$$\text{Thus: } L\frac{di}{dt}+Ri=L\times\left[\frac{E}{L}e^{-\frac{R}{L}t}\right]+R\left[\frac{E}{R}-\frac{E}{R}e^{-\frac{R}{L}t}\right]$$

$$\text{———} = \cancel{Ee^{-\frac{R}{L}t}}+E-\cancel{Ee^{-\frac{R}{L}t}}$$

$$\text{———} = E$$

$$\therefore\ L\frac{di}{dt}+Ri=E$$

HARDER EXPONENTIAL GROWTH AND DECAY

9 A vessel is filled at a variable rate so that the volume of liquid in the vessel at any time t is given by $V = A(1 - e^{-kt})$.

(a) Show that $\frac{dV}{dt} = k(A - V)$.

(b) If a quarter of the vessel is filled in the first 5 minutes, what fraction is filled in the next 5 minutes?

(c) Show that $\lim_{t \to +\infty} V = A$.

a) $\frac{dV}{dt} = -A \times (-k)\, e^{-kt} = Ak\, e^{-kt}$

whereas: $k(A - V) = k\left[A - A\left(1 - e^{-kt}\right)\right] = k\left[A e^{-kt}\right]$

So indeed $\frac{dV}{dt} = k(A - V)$

b) At $t = 0$ $\quad V = A\left(1 - e^{-k \times 0}\right) = 0$.

When $t \to +\infty$ $\quad V = A$ $\quad$ as $\lim_{t \to +\infty} e^{-kt} = 0$ if $k > 0$.

So the total volume of the vessel is A.

For $V = A/4$ (at $t = 5$) we must have $\frac{A}{4} = A\left(1 - e^{-5k}\right)$

$\Leftrightarrow \frac{1}{4} = 1 - e^{-k5} \Leftrightarrow e^{-5k} = 3/4 \Leftrightarrow \ln\left(e^{-5k}\right) = \ln(3/4)$

$\Leftrightarrow -5k = \ln(3/4)$ $\quad$ so $\quad k = \frac{1}{5}\ln(4/3)$

So $V = A\left[1 - e^{-\ln(4/3) \times t/5}\right] = A\left[1 - \left(\frac{4}{3}\right)^{-t/5}\right]$

So at $t = 10$ $\quad \frac{V}{A} = 1 - \left(\frac{4}{3}\right)^{-10/5} = 1 - \left(\frac{3}{4}\right)^2 = 7/16$

So in the next 5 minutes, the volume has increased from $\frac{1}{4}$ to $\frac{7}{16}$, so an extra $3/16$

c) $V(t) = A\left[1 - (4/3)^{-t/5}\right]$ $\quad$ so $\lim_{t \to +\infty} A\left[1 - (4/3)^{-t/5}\right] = A$

as $\lim_{t \to +\infty} (4/3)^{-t/5} = 0$

10 A rectangular vessel is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 20 cm, flows into the other compartment, initially empty, at a rate proportional to the difference between the levels in each compartment. The differential equation for this process is $\frac{dx}{dt} = k(20-2x)$, where x cm is the depth of the liquid in one of the vessels at any time t minutes.

(a) Show that $x = 10(1-e^{-2kt})$.

(b) If the level in the second compartment rises 2 cm in the first 5 minutes, at what time will the difference in levels be 2 cm?

a) $x(t) = 10\left(1 - e^{-2kt}\right) = 10 - 10e^{-2kt}$

So $\frac{dx}{dt} = -10 \times (-2k)e^{-2kt} = 20k\,e^{-2kt}$

whereas: $k(20-2x) = k\left[20 - 2\times 10\left(1-e^{-2kt}\right)\right] = k\left[\cancel{20} - \cancel{20} + 20e^{-2kt}\right]$
$= 20k\,e^{-2kt}$

So indeed $x(t) = 10\left(1-e^{-2kt}\right)$ satisfies $\frac{dx}{dt} = k(20-2x)$

b) At $t=5$ $x=2$

so $2 = 10 - 10e^{-2k\times 5} \iff 10e^{-10k} = 8 \iff e^{-10k} = \frac{4}{5}$

so $-10k = \ln(4/5)$ so $k = -\frac{1}{10}\ln(4/5)$

And $x(t) = 10\left[1 - e^{\ln(4/5)\times\frac{2t}{10}}\right] = 10\left[1 - (4/5)^{t/5}\right]$

For the difference to be 2 cm, we must have $x = 9$

$9 = 10\left[1-(4/5)^{t/5}\right] = 10 - 10(4/5)^{t/5} \iff 10(4/5)^{t/5} = 1$

$\iff \left(\frac{4}{5}\right)^{t/5} = \frac{1}{10} \iff \ln\left[\left(\frac{4}{5}\right)^{t/5}\right] = \ln(1/10) \iff \frac{t}{5}\ln(4/5) = \ln(1/10)$

so $t = 5\left[\frac{-\ln 10}{\ln 4 - \ln 5}\right] = \frac{5\ln 10}{\ln 5 - \ln 4} = 51$ min 35 seconds

(i.e. 51.594 minutes)

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11 In a certain chemical process, the amount y grams of a certain substance at time t hours is given by the formula $y = 3 + e^{-kt}$.

(a) Show that $\frac{dy}{dt} = -k(y-3)$.

(b) If initially y decreases at a rate of 0.08 grams per hour, find the value of k.

(c) Find the rate of change when $y = 3.5$. **(d)** What values can y take?

a) $y = 3 + e^{-kt}$ so $\frac{dy}{dt} = -k\,e^{-kt}$

whereas: $-k(y-3) = -k[\cancel{3} + e^{-kt} - \cancel{3}] = -k\,e^{-kt}$

so indeed $\frac{dy}{dt} = -k(y-3)$

b) At $t = 0$ $\frac{dy}{dt} = -k\,e^{-k\times 0} = -k$ so $k = 0.08$

c) so $y = 3 + e^{-0.08t}$

When $y = 3.5$ $\frac{dy}{dt} = -0.08(3.5-3)$ (as $\frac{dy}{dt} = -k(y-3)$)

so when $y = 3.5$, $\frac{dy}{dt} = -0.04$ g/hour.

d) When $t = 0$, $y(0) = 3 + e^{-k\times 0} = 4$

whereas $\lim_{t \to +\infty} y(t) = \lim_{t \to +\infty} [3 + e^{-kt}] = 3$

So y can take values between 3 and 4 grams.