CARTESIAN COORDINATES IN THREE-DIMENSIONAL SPACE

1 Find the lengths of the sides of the triangle ABC and determine whether it is isosceles, right-angled, equilateral or none of these.

(a) $A(5, 5, 1)$, $B(3, 3, 2)$, $C(1, 4, 4)$ **(b)** $A(3, 4, -1)$, $B(-5, 3, 0)$, $C(6, -7, 4)$

a) $AB = \sqrt{(5-3)^2+(5-3)^2+(1-2)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$BC = \sqrt{(3-1)^2+(3-4)^2+(2-4)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

$AC = \sqrt{(5-1)^2+(5-4)^2+(1-4)^2} = \sqrt{16+1+9} = \sqrt{26}$

So ABC is an isosceles triangle.

$(\sqrt{26})^2 = 26$ which is different of $3^2+3^2 = 18$

so it's not right-angled.

b) $AB = \sqrt{(3+5)^2+(4-3)^2+(-1-0)^2} = \sqrt{64+1+1} = \sqrt{66}$

$AC = \sqrt{(3-6)^2+(4+7)^2+(-1-4)^2} = \sqrt{9+121+25} = \sqrt{155}$

$BC = \sqrt{(-5-6)^2+(3+7)^2+4^2} = \sqrt{121+100+16} = \sqrt{237}$

So not isosceles or equilateral

$155+66 = 221 \neq 237$

so not right-angled either.

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2 Determine whether the given sets of points are collinear.

(a) $A(1, 2, 3)$, $B(3, 4, 1)$, $C(5, 6, -2)$ **(b)** $D(3, 1, -4)$, $E(-2, 3, 2)$, $F(1, 3, 5)$

a) $\overrightarrow{AB} = (3-1)\vec{i} + (4-2)\vec{j} + (1-3)\vec{k}$

$\overrightarrow{AB} = 2\vec{i} + 2\vec{j} - 2\vec{k}$

$\overrightarrow{AC} = (5-1)\vec{i} + (6-2)\vec{j} + (-2-3)\vec{k}$

$\overrightarrow{AC} = 4\vec{i} + 4\vec{j} - 5\vec{k}$

So as $\overrightarrow{AB} \neq k\,\overrightarrow{AC}$ then not colinear

b) $\overrightarrow{DE} = (-2-3)\vec{i} + (3-1)\vec{j} + (2+4)\vec{k}$

$\overrightarrow{DE} = -5\vec{i} + 2\vec{j} + 6\vec{k}$

$\overrightarrow{DF} = (1-3)\vec{i} + (3-1)\vec{j} + (5+4)\vec{k}$

$\overrightarrow{DF} = -2\vec{i} + 2\vec{j} + 9\vec{k}$

$\overrightarrow{DE} \neq k\,\overrightarrow{DF}$ with $k \in \mathbb{R}$.

so not colinear

CARTESIAN COORDINATES IN THREE-DIMENSIONAL SPACE

3 Find the equation of the sphere with centre C and radius r.

(a) $C(1, 2, 3)$, $r = 4$ **(b)** $C(-1, 0, 5)$, $r = \sqrt{3}$

a) $(x-1)^2+(y-2)^2+(z-3)^2=16$

$x^2-2x+1+y^2-4y+4+z^2-6z+9=16$

$x^2-2x+y^2-4y+z^2-6z=2$

b) $(x+1)^2+(y-0)^2+(z-5)^2=3$

$(x+1)^2+y^2+(z-5)^2=3$

$\boxed{\uparrow}$ **6** Find the equation of the sphere with centre $\overset{C}{(4, 5, -2)}$ that passes through the point $(1, 0, 0)$. O

$CO = \sqrt{(1-4)^2+5^2+(-2)^2} = \sqrt{9+25+4} = \sqrt{38}$

So the radius is $\sqrt{38}$

The equation is therefore:

$(x-4)^2+(y-5)^2+(z+2)^2=38$

4 Show that the given equation is the equation of a sphere and find the coordinates of its centre and the radius.

(a) $x^2 + y^2 + z^2 + 4x - 2y + 6z + 4 = 0$ **(b)** $x^2 + y^2 + z^2 - 8x + 4y + 6 = 0$

a) $x^2 + 4x + y^2 - 2y + z^2 + 6z = -4$

$(x+2)^2 - 2^2 + (y-1)^2 - 1 + (z+3)^2 - 9 = -4$

$(x+2)^2 + (y-1)^2 + (z+3)^2 = 10 = (\sqrt{10})^2$

So centre $(-2, 1, -3)$ and radius $\sqrt{10}$

b) $x^2 - 8x + y^2 + 4y + z^2 = -6$

$(x-4)^2 - 16 + (y+2)^2 - 4 + z^2 = -6$

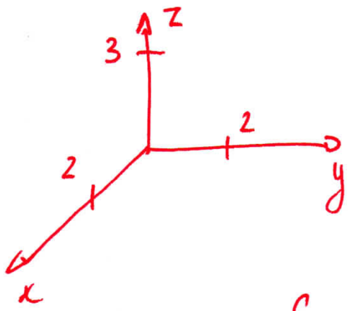
$(x-4)^2 + (y+2)^2 + z^2 = 14 = (\sqrt{14})^2$

So sphere of centre $(4, -2, 0)$ and radius $\sqrt{14}$

9 Find the equation of the sphere, centre (2, 2, 3) that:

(a) touches both the x–z and y–z planes **(b)** touches the x–y plane only.

a)



Distance to the $x-z$ plane is 2

———————— $y-z$ plane is 2

So radius 2

$$(x-2)^2+(y-2)^2+(z-3)^2=2^2$$

b) Distance to the $x-y$ plane is 3

So radius 3

$$(x-2)^2+(y-2)^2+(z-3)^2=3^2$$

11 The spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + (y-2)^2 + z^2 = 9$ intersect. Find:

(a) the value of y when they intersect

(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

a) $\begin{cases} x^2 + y^2 + z^2 = 16 \\ x^2 + (y-2)^2 + z^2 = 9 \end{cases}$

By elimination, we obtain $y^2 - (y-2)^2 = 16 - 9$

$$y^2 - (y^2 - 4y + 4) = 7$$

$$\Leftrightarrow \quad 4y - 4 = 7 \quad \Leftrightarrow \quad 4y = 11 \qquad \boxed{y = 11/4}$$

b) So $y = 11/4$

The points of sphere ① is $x^2 + \left(\frac{11}{4}\right)^2 + z^2 = 16$

$$x^2 + z^2 = 16 - \left(\frac{11}{4}\right)^2 = \frac{135}{16} \quad \text{and } y = 11/4$$

Centre $\left(0, \frac{11}{4}, 0\right)$ radius $\sqrt{\frac{135}{16}} = \frac{3\sqrt{15}}{4}$