FUNCTIONS AND RELATIONS

A function is a type of mathematical object that precisely describes a relationship between variables.

Definitions

- a real function f of a real variable x assigns to each element x of a given set of real numbers **exactly one** real number y , which is called the value of the function f at x . The dependence of y on f and x is made explicit by the notation $f(x)$, which means "the value of f at x. This is written $y = f(x)$
- the set of real numbers x on which f is defined is called the **domain** of f, while the set of values $f(x)$ obtained as x varies over the domain of f is called the **range** or image of f.
- the variable x is called the **independent variable**, as it may be chosen freely from the domain of f, while y is called the **dependent variable**, as its value depends on the particular value chosen for x
- a **function** may also be defined as a set of ordered pairs with the special property that no two pairs have the same first element (x value).

Finding the value of a function

Consider a function defined by the rule $f(x) = 2x - 7$. What is the value of the function when $x = 4$, $x = 0$ and $x = -1$? In the past you would have written the two equations $x = 4$, $y = 8 - 7 = 1$. With function notation, you can simply write $f(4) = 8 - 7 = 1$. Thus:

- $f(4) = 1$
- $f(0) = 0 - 7 = -7$
- $f(-1) = -2 - 7 = -9$

The function notation $f(x)$ allows you to write, in a single statement, the value of the independent variable as well as the corresponding dependent variable.

Example 1

Plot the following points on a number plane: $(-1,2)$, $(0,1)$, $(1,0)$, $(2,3)$, $(3,-2)$, $(4,0)$.

Does this set of points represent a function? Write its domain and range.

Solution

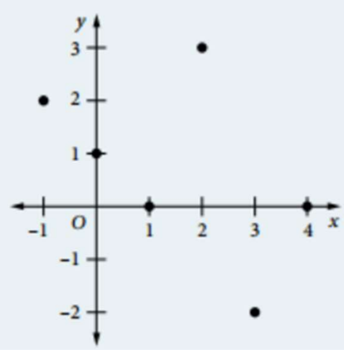
The graph shows that for each first value, x, there is only one second value, y.

This means the set of points is a function:

Domain = $\{-1, 0, 1, 2, 3, 4\}$

Range = $\{-2, 0, 1, 2, 3\}$

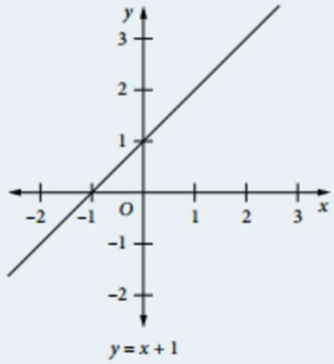
Note that 0 is the y value for two points, but it only needs to be listed once in the range.



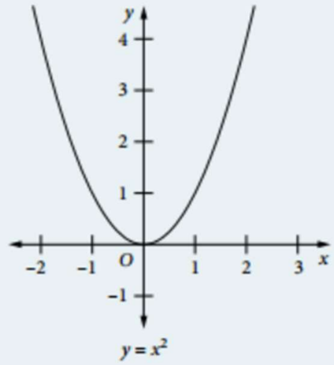
Example 2

Determine whether each graph represents a function. Write the domain and range for each.

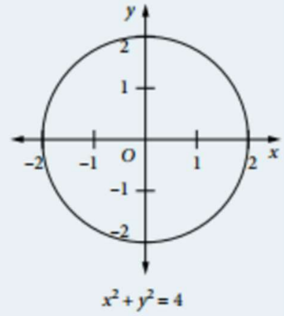
(a)



(b)



(c)



Solution

(a) Function

Domain: real numbers

Range: real numbers

Hence we can write $f(x) = x + 1$

(b) Function

Domain: real numbers

Range: real numbers, $y \geq 0$

Hence we can write $g(x) = x^2$

(c) Not a function

Domain: real numbers, $-2 \leq x \leq 2$

Range: real numbers, $-2 \leq x \leq 2$

In parts **(a)** and **(b)** the functions are different, so they have been given different labels, f and g.

Vertical line test

A simple way to determine whether a graph represents a function is to draw vertical lines. If each vertical line cuts the graph only once, then the graph represents a function. If any of the vertical lines cut the graph more than once, then the graph does not represent a function. The x values for which vertical lines do not cut the graph are not in the domain.

One-to-one functions

In a one-to-one function, every element in the range of the function corresponds to exactly one element in the domain.

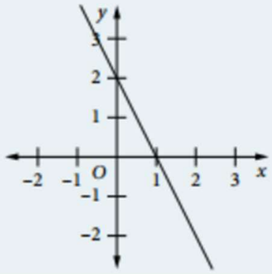
Horizontal line test

If a horizontal line intersects the graph of a function more than once then the function is not a one-to-one function. The graph of the parabola $y = x^2$ is an example of a function that can be intersected by a horizontal line.

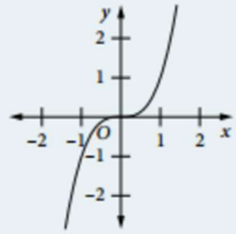
Example 3

Use the vertical line test to show that each graph represents a function. Write the domain and range for each.

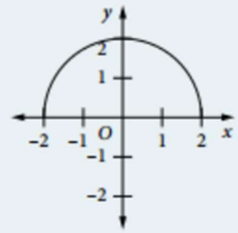
(a) $f(x) = 2 - 2x$



(b) $g(x) = x^3$



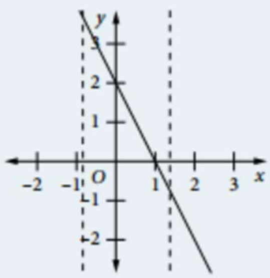
(c) $h(x) = \sqrt{4 - x^2}$



Solution

Draw several vertical lines on each graph.

(a)

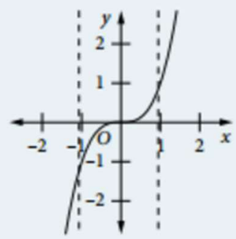


Vertical line can only cut once.

Domain: real values of x

Range: real values of y

(b)

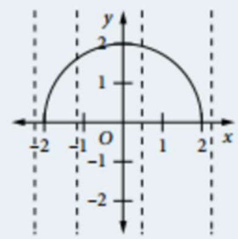


Vertical line can only cut once.

Domain: real values of x

Range: real values of y

(c)



Vertical line can only cut once, between $x = -2$ and $x = 2$.

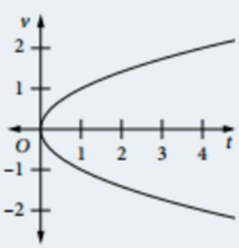
Domain: $-2 \leq x \leq 2$

Range: $0 \leq y \leq 2$

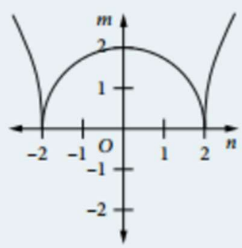
Example 4

Determine whether each graph represents a function or relation. Write the domain and range for each.

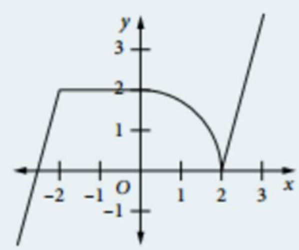
(a)



(b)



(c)



Solution

(a) Relation

Independent variable is t

Domain: real t, $t \geq 0$

Range: real numbers

(b) Function

Independent variable is n

Domain: real numbers

Range: real m, $m \geq 0$

(c) Function

Independent variable is x

Domain: real numbers

Range: real numbers

Types of functions and relations

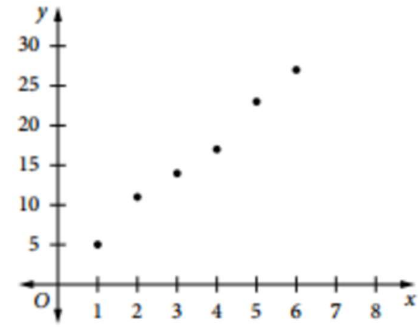
Sometimes other terms are used to define a function or relation in terms of a 'mapping' between two sets of numbers.

Consider the sets $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{5, 11, 14, 17, 23, 27\}$.

One-to-one functions

Make up the set of ordered pairs $\{(1, 5), (2, 11), (3, 14), (4, 17), (5, 23), (6, 27)\}$. This is a one-to-one mapping of set X onto set Y as each element of set X is paired with a different element of set Y. The resulting set of ordered pairs form a function as shown in the diagram.

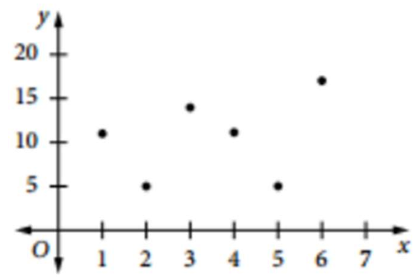
This function would pass the vertical line test.



Many-to-one functions

Make up the set of ordered pairs $\{(1, 11), (2, 5), (3, 14), (4, 11), (5, 5), (6, 17)\}$. This is a many-to-one mapping of set X onto set Y as more than one element of set X is paired with an element of set Y. The resulting set of ordered pairs form a function as shown in the diagram.

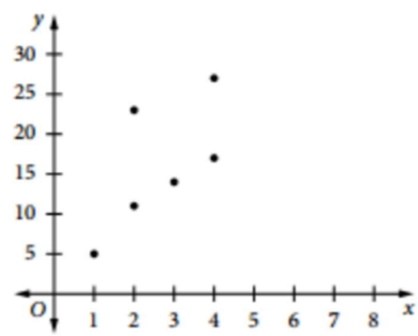
This function would pass the vertical line test, but it would not pass the horizontal line test as some y values are paired with more than one x value. This is why it is called 'many (x values) to one (y value)'.



One-to-many relations

Make up the set of ordered pairs $\{(1, 5), (2, 11), (3, 14), (4, 17), (2, 23), (4, 27)\}$. This is a one-to-many mapping of set X onto set Y as some elements of set X are paired with more than one element of set Y. The resulting set of ordered pairs is not a function, but a relation as shown in the diagram.

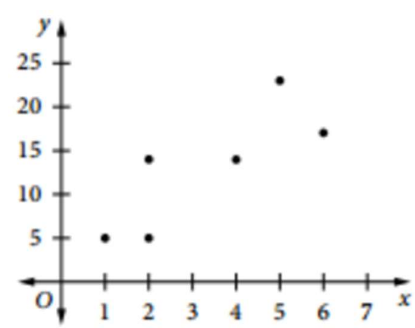
This set would not pass the vertical line test, but it would pass the horizontal line test as no y values occur more than once. This is why it is called 'one (y value) to many (x values)'.



Many-to-many relations

Make up the set of ordered pairs $\{(1, 5), (2, 5), (2, 14), (4, 14), (5, 23), (6, 17)\}$. This is a many-to-many mapping of set X onto set Y as some elements of set X are paired with more than one element of set Y and some elements of set Y are paired with more than one element of set X. The resulting set of ordered pairs form a relation as shown in the diagram.

This set would not pass the vertical line test nor the horizontal line test. Some y values are paired with more than one x value and some x values are paired with more than one y value. This is why it is called 'many (y values) to many (x values)'.



Function rules

When a function rule f is given and a domain is not specified, it is assumed that the domain of the function is the set of real numbers for which $f(x)$ defines a real number range. To find the domain, the solution of an inequality may be needed.

When the domain of f is all values of x over an interval, the graph of $y = f(x)$ is called the **curve** $y = f(x)$ and a part of the curve between two points is called an **arc**.

Example 5

State the largest possible domain for the function defined by the given rule. What is the range of each function?

(a) $f(x) = x^2$ **(b)** $f(x) = \frac{1}{x}$ **(c)** $f(x) = \sqrt{x}$ **(d)** $f(x) = \sqrt{4 - x^2}$ **(e)** $f(x) = \frac{x}{x^2 - 1}$

Solution

(a) Any real number squared is also a real number, so the domain of $f(x) = x^2$ is all real numbers.

Any real number squared is never negative, so the range of the function is all positive real numbers and zero.

(b) Fractions are not defined for a denominator of zero, so $\frac{1}{x}$ is defined for all values of x except $x = 0$. Thus the domain of $f(x) = \frac{1}{x}$ is all real numbers except $x = 0$. You can write $f(x) = \frac{1}{x}, x \neq 0$.

Because the numerator of $f(x)$ is never zero, we have $f(x) \neq 0$. The reciprocal of every non-zero real number is another non-zero real number, so the range of the function is all real numbers except zero.

(c) Only the square roots of non-negative numbers are real, so the domain of $f(x) = \sqrt{x}$ is real $x, x \geq 0$.

The square root of zero is zero and the square root of a positive real number is another positive real number, so the range of the function is all positive real numbers and zero: $f(x) \geq 0$.

(d) For the value of $f(x) = \sqrt{4 - x^2}$ to be real, $4 - x^2 \geq 0$, so $-2 \leq x \leq 2$. Therefore the domain of $f(x) = \sqrt{4 - x^2}$ is real $x, -2 \leq x \leq 2$ (or $|x| \leq 2$).

When $x = 0$, the value of the function is $f(0) = 2$; also, $f(2) = 0$ and $f(-2) = 0$. For all other values of x in the domain, $0 < f(x) < 2$, so the range of the function is the real numbers $0 \leq f(x) \leq 2$.

(e) The function is not defined when the denominator is zero, i.e. when $x^2 - 1 = 0$. This is true for $x = \pm 1$, so the domain of $f(x) = \frac{x}{x^2 - 1}$ is real $x, x \neq \pm 1$.

$f(0) = 0$, and for all values of x in the domain the function exists. The range of the function is the set of real numbers.

Example 6

A function is defined as $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2 - x, & x > 1 \end{cases}$. Find the domain and range of this function.

Solution

When $x \leq 1, f(x) = x^2$ exists for all real values of x.

When $x > 1, f(x) = 2 - x$ exists for all real values of x.

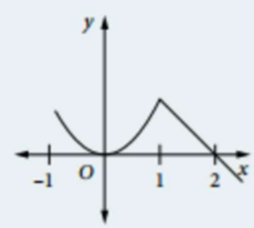
Thus the domain of the function is all real x.

When $x \leq 1, f(1) = 1, f(0) = 0, f(-1) = 1$ and $f(x) > 0$ when $x \neq 0$.

When $x > 1, f(1.01) = 0.99. f(2) = 0, f(3) = -1$ and for $x > 2, f(x) < 0$.

Thus the range of the function is all real numbers.

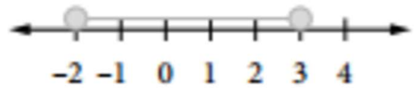
A sketch of the function shows this information more clearly.



FUNCTIONS AND RELATIONS

Set and Interval notations: are two different ways of representing an interval between two real values. These notations are explained through examples:

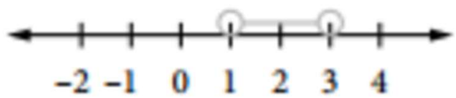
Example 1: for the values between (-2) and $(+3)$, both these values <u>included</u>, as shown below:



As both values (-2) and $(+3)$ are both included, this is called a "**closed**" interval.

set notation	interval notation
$-2 \leq x \leq 3$	$x \in [-2, 3]$
• the signs $\leq$ means "*less than or equal to*"	• the sign "$\in$" means "*belongs to*" • the square bracket [pointing inwards towards (-2) shows that this value is included in the interval. • the square bracket] pointing inwards towards 3 shows that this value is included in the interval.

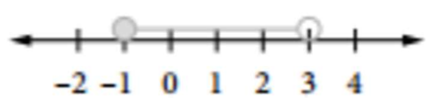
Example 2: for the values between (-1) (included) and $(+3)$ (not included) as shown below:



As both of the values $(+1)$ and $(+3)$ are both excluded, this is called a "**open**" interval.

set notation	interval notation
$1 < x < 3$	$x \in (1, 3)$
• the signs $<$ means "*strictly less than*"	• the round bracket (pointing inwards towards 1 shows that this value is NOT included from the interval. • the round bracket) pointing inwards towards 3 shows that this value is also NOT included.

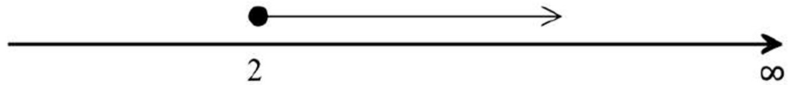
Example 3: for the values between (-1) (included) and $(+3)$ (not included) as shown below:



As only one of the values (-1) and $(+3)$ is included, this is called a "half-open" interval.

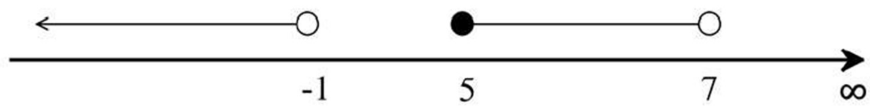
set notation	interval notation
$-1 \leq x < 3$	$x \in [-1, 3)$
	• the square bracket [pointing inwards towards (-1) shows that this value is included. • the round bracket) pointing inwards towards 3 shows that this value is NOT included.

Example 4: for the values greater than $(+2)$ (included) as shown on the number line below:



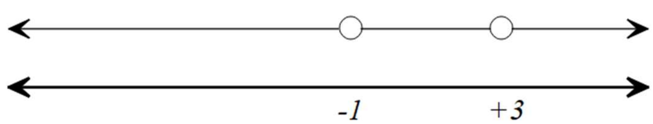
set notation	interval notation
$x \geq 2$	$x \in [2, +\infty)$
	• a round bracket) is used to the right of the interval to reflect that one can always find a number greater than any large number, so it is open towards $+\infty$

Example 5: for the values less than (-1) (excluded), and between 5 (included) and 7 (excluded), as shown below:



set notation	interval notation
$x < 1$ and $5 \leq x < 7$	$x \in (-\infty, 1) \cup [5, 7)$
	• again, a round bracket (is used to the left of the interval to reflect that one can always find a number smaller than any small number, so it is open towards $-\infty$ • the sign $\cup$ means "*union*", to show that x belong to the union of two separate intervals, namely $(-\infty, 1)$ and $[5, 7)$

Example 6: for the values shown on the number line below:



set notation	interval notation
$x \in \mathbb{R} - \{-1, 3\}$	$x \in (-\infty, -1) \cup (-1, +3) \cup (+3, +\infty)$
• the symbol $\mathbb{R}$ represents the set of real numbers[1] • so all values of $\mathbb{R}$ are included except the two values (-1) and $+3$	

[1] "*real numbers*" are all numbers that you know so far, decimal or integer. In the Extension 2 course, we study another set of numbers called "*complex numbers*" which square is negative, such as the number i, i.e. $i^2 = -1$. These complex numbers do not exist in real life, however, turn out to be very useful particularly in Physics.