CUBIC POLYNOMIALS

So far you have considered the linear polynomial, $ax + b$ and the quadratic polynomial $ax^2 + bx + c$. The graph of $y = ax + b$ is a straight line and the graph of $y = ax^2 + bx + c$ is a parabola. The equation $ax + b = 0$ has one root whilst the equation $ax^2 + bx + c = 0$ has at most two real roots, $a \neq 0$ in each case.

These functions are continuous: they have no gaps in them over their domain. Simply put, this means that they could be drawn without taking your pen off the paper. In particular, all polynomial functions are continuous.

The general polynomial of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ is a polynomial of degree n, n a positive integer, $a_n \neq 0$, with coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$. a_0 is called the constant term.

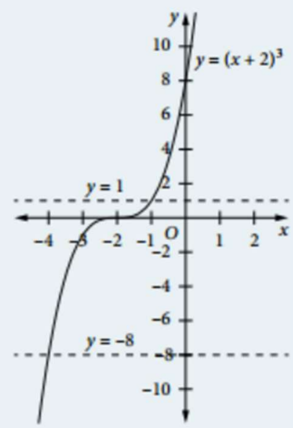
The cubic polynomial is of the form $ax^3 + bx^2 + cx + d$. The cubic function is easy to sketch without technology when it can be written in any of the forms $y = kx^3$, $y = k(x - b)^3 + c$ or $y = k(x - a)(x - b)(x - c)$ where a, b, c and k are constants, $k \neq 0$.

A cubic polynomial is of degree 3, that is the largest power of x is x^3. In $ax^3 + bx^2 + cx + d$, ax^3 is called the leading term and a is called the coefficient of the leading term. b and c are coefficients and d is called the constant term.

Example 19

Draw the graph of $y = (x + 2)^3$. On your graph, draw the lines $y = 1$ and $y = -8$. Use the graph to solve the equations:

(a) $(x + 2)^3 = 0$ **(b)** $(x + 2)^3 = -8$ **(c)** $(x + 2)^3 = 1$



Solution

(a) $(x + 2)^3 = 0$ where the graph cuts the x-axis, so $x = -2$ is the solution.

(b) $(x + 2)^3 = -8$ where the graph cuts the line $y = -8$, so $x = -4$ is the solution.

(c) $(x + 2)^3 = 1$ where the graph cuts the line $y = 1$, so $x = -1$ is the solution.

These equations could also have been solved algebraically. In each case, take cube root of both sides:

(a) $(x + 2)^3 = 0$, $x + 2 = 0$, $x = -2$

(b) $(x + 2)^3 = -8$, $x + 2 = -2$, $x = -4$

(c) $(x + 2)^3 = 1$, $x + 2 = 1$, $x = -1$

You may know the terms 'turning point', 'local maximum' and 'local minimum'.

The term **turning point** describes itself. It is the point where the curve turns around, i.e. stops rising and starts to fall or stops falling and starts to rise. The curve in Example 20 has a turning point between $x = -1$ and $x = 0$ and another between $x = 1$ and $x = 2$. The first is a maximum turning point (or **local maximum**); the second is a minimum turning point (or **local minimum**).

At a maximum turning point a curve is concave down; at a minimum turning point a curve is concave up. Between two points like this, there must be a point where the concavity changes. This is called a **point of inflection** and will be discussed more fully later in the course.

In Example 19, the graph of $y = (x + 2)^3$ does not have any turning points as it is always sloping up to the right. For this curve, $x = -2$ gives the point of inflection as it is clear that the concavity changes from concave down to concave up at the point $(-2, 0)$.

Example 20

Draw the graph of $y = (x+1)(x-1)(x-2)$, using technology if necessary. On your graph, draw the lines $y = 2$ and $y = -4$. Use the graph to solve the following equations.

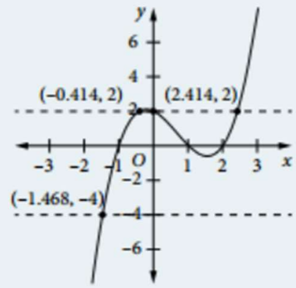
(a) $(x+1)(x-1)(x-2) = 0$ **(b)** $(x+1)(x-1)(x-2) = 2$ **(c)** $(x+1)(x-1)(x-2) = -4$

Solution

(a) $(x+1)(x-1)(x-2) = 0$ where the graph cuts the x-axis, so $x = -1, 1, 2$

(b) $(x+1)(x-1)(x-2) = 2$ where the graph cuts the line $y = 2$, so $x = -0.414, 0, 2.414$ (3 d.p.)

(c) $(x+1)(x-1)(x-2) = -4$ where the graph cuts the line $y = -4$, so $x = -1.468$ (3 d.p.)



These equations could also have been solved algebraically. It is easy to solve **(a)** $(x+1)(x-1)(x-2) = 0$ algebraically because it is in factored form.

To solve the other two equations algebraically, you need to expand the left-hand side and collect like terms. Part **(b)** is then solvable as the constant term disappears, but in part **(c)** the result is still a cubic and not factorised. Using graphing software is often the only easy way to solve a cubic equation like this.

Example **20** shows how horizontal lines $y = c$ drawn on a cubic graph can be used to see how many solutions there will be. If the line $y = c$ is drawn on the graph in Example **20**, then the graph shows that the equation $(x+1)(x-1)(x-2) = c$ would always have at least one solution. If $y = c$ touches the curve at the local maximum or minimum value, it will cut the curve again so that the equation $(x+1)(x-1)(x-2) = c$ will have two distinct solutions.
If $y = c$ cuts the curve between the local maximum or minimum values, the equation $(x+1)(x-1)(x-2) = c$ will have three distinct solutions. If $y = c$ cuts the curve above the local maximum or below the local minimum value, the equation $(x+1)(x-1)(x-2) = c$ will only have one solution.

Example 21

Draw the graph of $y = x^3 - 2x^2 - 3x + 4$. By drawing appropriate lines on your graph, use this to solve:

(a) $x^3 - 2x^2 - 3x + 4 = 0$ **(b)** $x^3 - 2x^2 - 3x + 4 = 4$ **(c)** $x^3 - 2x^2 - 3x + 4 = -3$

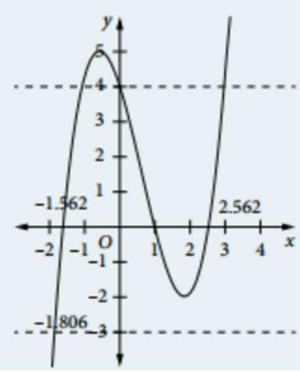
Solution

Use graphing software to obtain this graph. Software will also give the points of intersection.

(a) Use the x-axis (for the line $y = 0$) and find the intersections: $x = -1.562, 1, 2.562$. This means the equation has a factor of $(x - 1)$.

(b) Draw $y = 4$ and find the intersections: $x = -1, 0, 3$. This means that the equation can be factorised as $x(x+1)(x-3) = 0$.

(c) Draw $y = -3$ and find the intersections: $x = -1.806$



Example 22

Draw the graph of $y = (x+2)^2(1-x)$. By drawing appropriate lines on your graph, use this to solve the following:

(a) $(x+2)^2(1-x) = 0$ **(b)** $(x+2)^2(1-x) = 4$ **(c)** $(x+2)^2(1-x) = -2$

(d) For what values of c will the equation $(x+2)^2(1-x) = c$ have three distinct roots?

(e) What is the coefficient of x^3 when $(x+2)^2(1-x)$ is expanded?

Solution

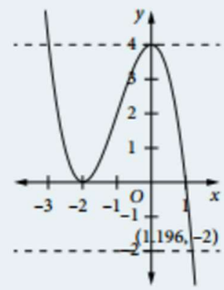
(a) Find where the graph cuts the x-axis: $x = -2, 2$.
Note that $(x+2)^2 = 0$ gives a single value of x.

(b) Cuts the line $y = 4$: $x = -3, 0$.

(c) Cuts the line $y = -2$: $x = 1.196 \approx 1.2$

(d) There will be three distinct roots where a horizontal line cuts the graph three times. This happens only between the local minimum and local maximum, so: $0 < c < 4$

(e) $(x+2)^2(1-x) = -x^3 - 3x^2 + 4$. The coefficient of x^3 is -1.



Note that as $x \to \infty$, $y \to -\infty$ and that the coefficient of x^3 is -1.

Properties of the cubic polynomial

For the polynomial $y = ax^3 + bx^2 + cx + d$, $a \neq 0$:

- For $a > 0$, if $x \to \infty$ then $y \to ax^3$ and $y \to \infty$
  if $x \to -\infty$ then $y \to -ax^3$ and $y \to -\infty$
- For $a < 0$, if $x \to \infty$ then $y \to ax^3$ and $y \to -\infty$
  if $x \to -\infty$ then $y \to -ax^3$ and $y \to \infty$
- The graph will cut the x-axis at least once;
  $ax^3 + bx^2 + cx + d = 0$ for at least one value of x.

Cubic equations

The general cubic function is written $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. In this course you will look at the simpler versions of the cubic function given by $y = k(x-b)^3 + c$, $k \neq 0$ and $y = k(x-a)(x-b)(x-c)$, $k \neq 0$.

You can show by expanding and collecting like terms that:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Note that these results are quite different from the factorisation of $a^3 - b^3$ and $a^3 + b^3$ studied in Chapter 1.

Using the expansion of $(x-b)^3$ you can show that $a(x-b)^3 + c$ becomes an expression of the form $ax^3 + bx^2 + cx + d$, $a \neq 0$.

Solving equations of the type $k(x-b)^3+c=0$

Example 23

Solve the equation $2(x-1)^3+5=0$.

Solution

Rearrange the equation:

$$2(x-1)^3+5=0$$
$$2(x-1)^3=-5$$
$$(x-1)^3=-\frac{5}{2}$$

Take the cube root of both sides:
(You can always find the odd root of a negative number.)

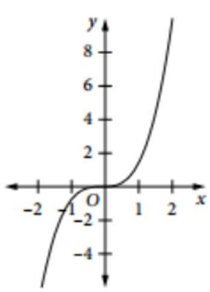
$$x-1=-\sqrt[3]{\frac{5}{2}}$$
$$x=1-\sqrt[3]{\frac{5}{2}}$$

To put this in standard form required for surds (that is, to rationalise the denominator), multiply the numerator and denominator of the fraction by $\sqrt[3]{4}$:

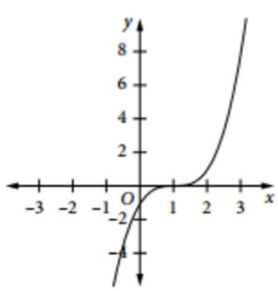
$$x=1-\frac{\sqrt[3]{5}}{\sqrt[3]{2}}\times\frac{\sqrt[3]{4}}{\sqrt[3]{4}}$$
$$x=1-\frac{\sqrt[3]{20}}{\sqrt[3]{8}}$$
$$x=1-\frac{\sqrt[3]{20}}{2}$$

Even though the equation in Example 23 is cubic, it only has one root. This becomes obvious when you consider the following graphs.

$y=x^3$

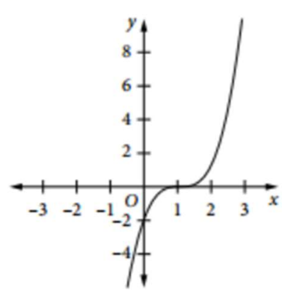


$y=(x-1)^3$



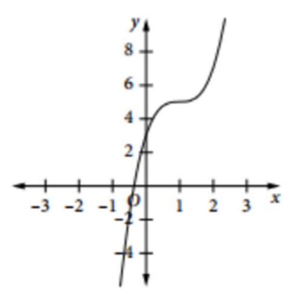
This is the graph of $y=x^3$ moved 1 unit to the right.

$y=2(x-1)^3$



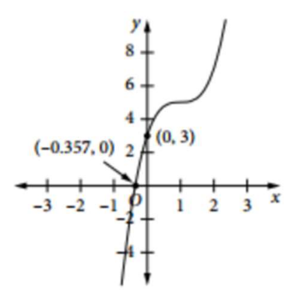
This is the graph of $y=(x-1)^3$ stretched vertically by a factor of 2.

$y=2(x-1)^3+5$



This is the graph of $y=(x-1)^3$ stretched vertically by a factor of 2 and moved upwards 5 units.

Graphically, it is easy to see the solution to $x^3=0$, $(x-1)^3=0$ and $2(x-1)^3=0$. It is not so easy to see the solution to $2(x-1)^3+5=0$. An approximate solution may be found using graphing software and clicking on the point of intersection of the graph and the x-axis, and reading off the x-value at this point.



The diagram shows that the graph cuts the x-axis at $(-0.357, 0)$ so the solution to $2(x-1)^3+5=0$ is $x=-0.357$. This is a good approximation to the exact value of the root found in Example 23.

In graphing each of these functions notice how the curve flattens out at $x=0$ for $y=x^3$ and at $x=1$ for the other graphs. It would appear that the slope of the curve is zero at this point. It is also worth noting that at all other points on the curve the slope is positive. The significance of these observations will become clear later, after you have learnt how to find the derivative of a function.