PARAMETRIC AND CARTESIAN EQUATIONS

The position of any point P, relative to an origin O, is uniquely specified by the vector $\overline{OP}$, which is termed the *position vector* of P (relative to O).

Consider a body moving in the $\underset{\sim}{i}-\underset{\sim}{j}$ plane so that its position at time, t, is given by the position vector $\underset{\sim}{r}(t) = (3-t)\underset{\sim}{i} + 2t\underset{\sim}{j}$, $t \geq 0$.

By substituting values of t into $\underset{\sim}{r}(t)$, you can determine where the body is at these times.

For example:

t	$\underset{\sim}{r}(t)$	position
0	$3\underset{\sim}{i}$	(3, 0)
0.5	$2.5\underset{\sim}{i} + \underset{\sim}{j}$	(2.5, 1)
1	$2\underset{\sim}{i} + 2\underset{\sim}{j}$	(2, 2)
2	$\underset{\sim}{i} + 4\underset{\sim}{j}$	(1, 4)

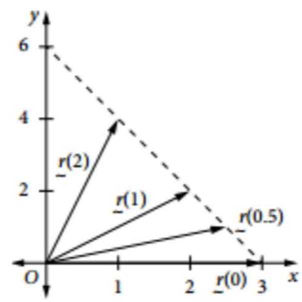
The figure on the right shows these four position vectors and the path that the body appears to be taking.

The graph of a vector equation is the set of points determined by $\underset{\sim}{r}(t)$ as t varies.

For $t \geq 0$, $\underset{\sim}{r}(0)$ gives us the initial position of the body.

Given that $\underset{\sim}{r}(0) = 3\underset{\sim}{i}$, you conclude that the body begins its motion at (3, 0).

Substituting increasing values of t into $\underset{\sim}{r}(t)$ can help you determine the body's direction of motion.

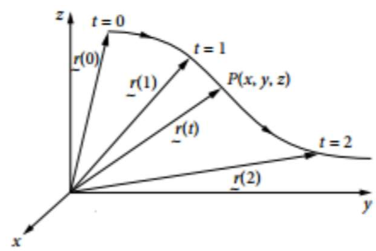


The body starts at (3, 0) and appears to be moving along the line with equation $y = 6 - 2x$.

In general, when a body moves in space, its position vector varies with time.

Having the vector equation of the paths means that you are only working with one variable, t, rather than three variables x, y and z.

This is illustrated in the diagram to the right where the position vector is denoted by $\underset{\sim}{r}(t)$ to indicate its dependence on time, t.



If the coordinates of the variable point $P(x, y, z)$ are expressed parametrically as $x = f(t)$, $y = g(t)$ and $z = h(t)$, then the **vector equation** of the path is given by the position vector $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x-, y- and z-axes respectively.

The equations $x = f(t)$, $y = g(t)$ and $z = h(t)$, are called **parametric equations** and t is called the **parameter**.

In the introductory example, the parametric equations are $x = 3 - t$ and $y = 2t$.

Rearranging the first equation to make the parameter t the subject gives $t = 3 - x$.

Substituting $t = 3 - x$ into $y = 2t$ gives $y = 2(3 - x) = 6 - 2x$.

So, confirming the earlier assertion, $y = 6 - 2x$ is the Cartesian equation that describes the path of the body.

Example 20

State the vector equation of a curve that has parametric equations given by $x = 2\cos t$, $y = 2\sin t$ and $z = 1$ for $t \geq 0$.

Solution

The general form is $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$.

The vector equation of the curve is $\underset{\sim}{r}(t) = (2\cos t)\underset{\sim}{i} + (2\sin t)\underset{\sim}{j} + \underset{\sim}{k}$, $t \geq 0$.

Example 21

The position vector of a body at time t is given by $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.

(a) Find the Cartesian equation of the path of the body and state the domain.

(b) Sketch the path of the body.

Solution

(a) State and number the parametric equations: $x = 1 - t$ [1]

$y = t^2$ [2]

Express t in terms of x, from [1]: $t = 1 - x$

State the allowed values of x: If $t \geq 0$, then $x \leq 1$.

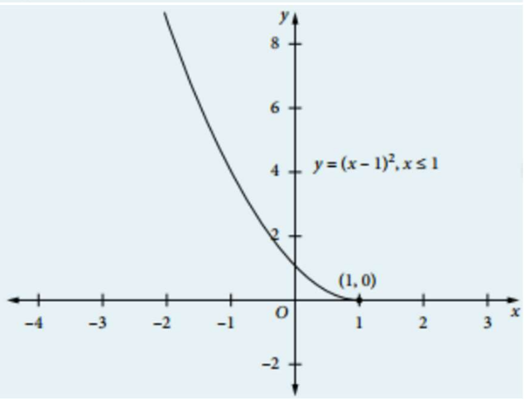
Substitute into [2] to eliminate t: $y = (1-x)^2$

State the Cartesian equation with domain: $y = (1-x)^2$, $x \leq 1$.

(b) Sketch the parabola with correct domain:
The path is that of the parabola $y = (1-x)^2$ for which $x \leq 1$.

To summarise from part **(a)**:

1. The parametric equations are $x = 1 - t$ and $y = t^2$, $t \geq 0$.
2. The Cartesian equation is $y = (1-x)^2$, $x \leq 1$.
3. The vector equation is $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.

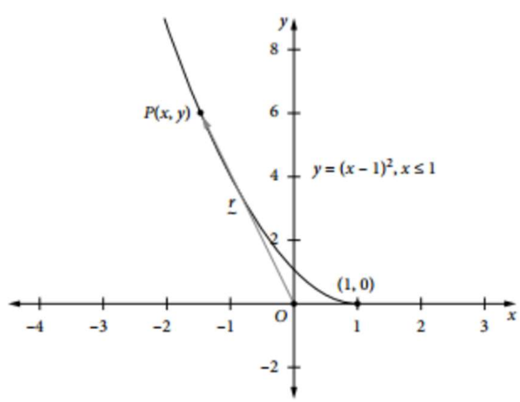


As shown in Example **21**, elimination of the parameter t gives the Cartesian equation of the body's path shown in the figure on the right.

The vector equation $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$ is an example of a **vector function of the scalar** (real) variable t because each element in the domain gives a unique value of $\underset{\sim}{r}$.

Consequently, the function defined by $\underset{\sim}{r}$ is a **one-to-one** function.

To specify a vector function, it is sufficient to state its rule and the domain, i.e. $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.



Example 22

The parametric equations of a curve are $x = \cos 2t$, $y = \sin t$, $t \geq 0$.

(a) Find the Cartesian equation of the curve. **(b)** Find the vector equation of the curve.

(c) Sketch the curve.

Solution

(a) $x = \cos 2t$, $y = \sin t$, $t \geq 0$

$x = \cos 2t$

$= 1 - 2\sin^2 t$

$= 1 - 2y^2$

$2y^2 = 1 - x$

$y^2 = \frac{1-x}{2}$, $-1 \leq x \leq 1$ is the Cartesian equation of the curve.

(b) $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$\underset{\sim}{r} = (\cos 2t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j}$, $t \geq 0$ is the vector equation of the curve.

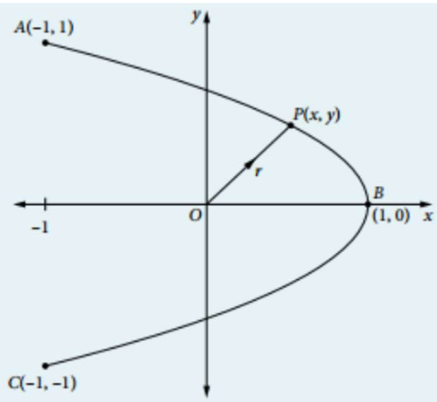
(c) Since $-1 \leq \cos 2t \leq 1$ for all t, it follows that $-1 \leq x \leq 1$ and so the equation represents only the arc ABC of the parabola as shown in the diagram.

When $t = 0$, $x = 1$, $y = 0$ and $\underset{\sim}{r} = \underset{\sim}{i}$.

When $t = \frac{\pi}{2}$, $x = -1$, $y = 1$ and $\underset{\sim}{r} = -\underset{\sim}{i} + \underset{\sim}{j}$.

When $t = \pi$, $x = 1$, $y = 0$ and $\underset{\sim}{r} = \underset{\sim}{i}$.

When $t = \frac{3\pi}{2}$, $x = -1$, $y = -1$ and $\underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{j}$.



From this you can see that if you consider the curve as a path traced out by a moving point, the point starts at B, moves to A, then back to B, then to C and back to B as t increases from 0 to 2π. Since x and y are periodic functions of t, the curve retraces itself as t increases beyond 2π.