DERIVATIVE OF $f(x)=e^x$ AND $f(x)=e^{kx}$

1 Write the derivative of:

(a) e^{4x} **(b)** $2e^{\frac{x}{2}}$ **(c)** $e^{4x}-e^{3x}$ **(d)** $2e^{3x}+e^{-x}$ **(e)** $4e^{3x}-e^{-2x}$

a) $f(x) = e^{4x}$ so using the chain rule $f'(x) = 4e^{4x}$

b) $f(x) = 2e^{x/2}$ so $f'(x) = 2 \times \frac{1}{2} e^{x/2} = e^{x/2}$

c) $f(x) = e^{4x} - e^{3x}$ so $f'(x) = 4e^{4x} - 3e^{3x}$

d) $f(x) = 2e^{3x} + e^{-x}$ so $f'(x) = 2 \times 3 e^{3x} + (-1) e^{-x}$

$f'(x) = 6e^{3x} - e^{-x}$

e) $f(x) = 4e^{3x} - e^{-2x}$

so $f'(x) = 12e^{3x} + 2e^{-2x}$

DERIVATIVE OF $f(x)=e^x$ AND $f(x)=e^{kx}$

2 If $y = e^{x^2}$ then $\frac{dy}{dx}$ is: **A** $x^2e^{x^2}$ $\boxed{\textbf{B}}$ $2xe^{x^2}$ **C** x^2e^{2x} **D** $2xe^{2x}$

$\frac{dy}{dx} = e^{x^2} \times 2x = 2x\, e^{x^2}$ so $\boxed{B}$

3 Differentiate:

(a) x^2e^{3x} **(b)** $(2x+1)e^{-x}$ **(c)** $(x^2+x+1)e^{2x}$ **(d)** xe^{-2x}

We use the product rule, and the chain rule

a) $f(x) = x^2 e^{3x} = u(x) \times v(x)$ with $u(x) = x^2$ and $v(x) = e^{3x}$

$u'(x) = 2x$ $v'(x) = 3e^{3x}$ so $f'(x) = 2x\,e^{3x} + x^2 \times 3e^{3x}$

$f'(x) = xe^{3x}[3x+2]$

b) $f(x) = (2x+1)e^{-x}$ $u(x) = 2x+1$ $u'(x) = 2$

$v(x) = e^{-x}$ $v'(x) = -e^{-x}$

$f'(x) = 2e^{-x} - (2x+1)e^{-x} = e^{-x}[1-2x]$

c) $f(x) = (x^2+x+1)e^{2x}$ $u(x) = x^2+x+1$ $u'(x) = 2x+1$

$v(x) = e^{2x}$ $v'(x) = 2e^{2x}$

$f'(x) = (2x+1)e^{2x} + 2(x^2+x+1)e^{2x} = e^{2x}[2x^2+4x+3]$

d) $f(x) = x e^{-2x} = u(x)\, v(x)$

$u(x) = x$ $u'(x) = 1$

$v(x) = e^{-2x}$ $v'(x) = -2e^{-2x}$

$f'(x) = 1 \times e^{-2x} - 2e^{-2x} \times x$

$f'(x) = e^{-2x}[-2x+1]$

DERIVATIVE OF $f(x)=e^x$ AND $f(x)=e^{kx}$

(i) $\frac{e^{3x}}{x}$ (j) $\frac{x^3}{e^x}$ (k) $\frac{e^{4x}}{x-1}$ (l) $\frac{e^x}{\sqrt{x}}$

i) $f(x) = \frac{e^{3x}}{x} = \frac{u(x)}{v(x)}$ $u(x) = e^{3x}$ $u'(x) = 3e^{3x}$

$v(x) = x$ $v'(x) = 1$

Quotient rule

$$f'(x) = \frac{3e^{3x} \times x - 1 \times e^{3x}}{x^2} = \frac{e^{3x}[3x-1]}{x^2}$$

j) $f(x) = \frac{x^3}{e^x} = \frac{u(x)}{v(x)}$ quotient rule $u(x) = x^3$ $u'(x) = 3x^2$

$v(x) = e^x$ $v'(x) = e^x$

$$f'(x) = \frac{3x^2 \times e^x - e^x \times x^3}{(e^x)^2} = \frac{e^x}{(e^x)^2}[3x^2 - x^3] = \frac{x^2(3-x)}{e^x}$$

k) $f(x) = \frac{e^{4x}}{x-1} = \frac{u(x)}{v(x)}$ $u(x) = e^{4x}$ $u'(x) = 4e^{4x}$

$v(x) = x-1$ $v'(x) = 1$

$$f'(x) = \frac{4e^{4x} \times (x-1) - 1 \times e^{4x}}{(x-1)^2} = \frac{e^{4x}[4x-4-1]}{(x-1)^2} = \frac{e^{4x}[4x-5]}{(x-1)^2}$$

e) $f(x) = \frac{e^x}{\sqrt{x}} = \frac{u(x)}{v(x)}$ $u(x) = e^x$ $u'(x) = e^x$

$v(x) = \sqrt{x}$ $v'(x) = \frac{1}{2\sqrt{x}}$

so $$f'(x) = \frac{e^x \times \sqrt{x} - \frac{1}{2\sqrt{x}} e^x}{(\sqrt{x})^2}$$

$$f'(x) = e^x \left[\frac{\sqrt{x} - \frac{1}{2\sqrt{x}}}{x}\right] = e^x \left[\frac{2x-1}{2x\sqrt{x}}\right]$$

DERIVATIVE OF $f(x)=e^x$ AND $f(x)=e^{kx}$

4 Differentiate:

(a) e^{2x+3} **(b)** e^{x^2-2x} **(c)** $3e^{-x^3}$ **(d)** $2e^{3x-1}$ **(e)** $e^{3x-1}+e^{4x+2}$ **(f)** $\sqrt{x}e^{-x}$ **(g)** $3e^{2x^2}$ **(h)** $3e^{2x-1}$ **(i)** xe^{x^2}

a) $f(x) = e^{2x+3}$ $\quad f'(x) = 2e^{2x+3}$ (chain rule)

b) $f(x) = e^{x^2-2x}$ $\quad f'(x) = e^{x^2-2x} \times (2x-2) = 2(x-1)e^{x^2-2x}$

c) $f(x) = 3e^{-x^3}$ $\quad f'(x) = 3e^{-x^3} \times (-3x^2) = -9x^2e^{-x^3}$

d) $f(x) = 2e^{3x-1}$ $\quad f'(x) = 2e^{3x-1} \times 3 = 6e^{3x-1}$

e) $f(x) = e^{3x-1} + e^{4x+2}$ $\quad f'(x) = 3e^{3x-1} + 4e^{4x+2}$

f) $f(x) = \sqrt{x}\,e^{-x} = u(x)\,v(x)$ (product rule)

$u(x) = \sqrt{x}$ $\quad u'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad f'(x) = \dfrac{1}{2\sqrt{x}} \times e^{-x} - \sqrt{x}\,e^{-x} = e^{-x}\left[\dfrac{1}{2\sqrt{x}} - \sqrt{x}\right]$

$v(x) = e^{-x}$ $\quad v'(x) = -e^{-x}$

g) $f(x) = 3e^{2x^2}$ $\quad f'(x) = 3e^{2x^2} \times 4x = 12xe^{2x^2}$

h) $f(x) = 3e^{2x-1}$ $\quad f'(x) = 6e^{2x-1}$

i) $f(x) = xe^{x^2}$ product rule

$u(x) = x$ $\quad u'(x) = 1$

$v(x) = e^{x^2}$ $\quad v'(x) = 2xe^{x^2}$

so $f'(x) = 1 \times e^{x^2} + 2xe^{x^2} \times x$

$f'(x) = e^{x^2}\left[2x^2+1\right]$

6 If $x = (1 + t)e^{5t}$, prove that $\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 25x = 0$.

$u(t) = 1+t \qquad u'(t) = 1$

$v(t) = e^{5t} \qquad v'(t) = 5e^{5t}$

product rule

$$\frac{dx}{dt} = 1 \times e^{5t} + 5e^{5t}(1+t)$$

so $\frac{dx}{dt} = e^{5t}[5t+6]$

we differentiate again to find $\frac{d^2x}{dt^2}$, using the product rule again

$u(t) = e^{5t} \qquad u'(t) = 5e^{5t}$

$v(t) = 5t+6 \qquad v'(t) = 5$

So $\frac{d^2x}{dt^2} = 5e^{5t}(5t+6) + 5e^{5t} = e^{5t}[25t+35]$

Now we calculate $\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 25x$.

$$\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 25x = e^{5t}[25t+35] - 10e^{5t}[5t+6] + 25[1+t]e^{5t}$$

$$= e^{5t}[t\underbrace{(25-50+25)}_{=0} + \underbrace{(35-60+25)}_{=0}]$$

so indeed $\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 25x = 0$

9 Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = 1$. Find also the coordinates of the points where the tangent intersects: **(a)** the x-axis **(b)** the y-axis.

At $x=1$ $f(1)=e^{2\times 1}=e^2$

so the curve and its tangent pass through the point $(1, e^2)$

$f'(x)=2e^{2x}$ so the gradient at $x=1$ is $2e^{2\times 1}=2e^2$

$\therefore$ the equation of the tangent at $x=1$ is

$$y-e^2=2e^2(x-1) \iff y=2e^2x-e^2$$

a) The tangent $y=2e^2x-e^2$ intercepts the x-axis when $y=0$, i.e when $2e^2x-e^2=0$

$\iff e^2[2x-1]=0$ so at $x=\frac{1}{2}$

$\therefore$ Point $\left(\frac{1}{2}, 0\right)$

b) it intercepts the y-axis when $x=0$, i.e. $y=0-e^2$

so at the point $(0, -e^2)$

DERIVATIVE OF $f(x)=e^x$ AND $f(x)=e^{kx}$

10 Write the equation of the tangent and the normal to the curve $y = 2 + e^{-x}$ at the point where $x = 0$.

at $x=0$ $f(0) = 2 + e^{-0} = 2 + 1 = 3$ so point $(0,3)$.

$f'(x) = -e^{-x}$ so $f'(0) = -e^{-0} = -1$

So the tangent is $y - 3 = -(x-0) \Leftrightarrow y = -x + 3$

And the normal is $y - 3 = +\frac{1}{1}(x-0)$

OR $y = x + 3$

11 After n years, the value V of a principal of P dollars that is invested at a rate of r% per year (with r expressed as a decimal) and compounded continuously is given by $V = Pe^{rn}$. Show that $\frac{dV}{dn} = Vr$.

$V(n) = Pe^{rn}$ where P and r are constants.

$\frac{dV}{dn} = P \times r e^{rn}$

So $\frac{dV}{dn} = r \times Pe^{rn} = rV$

So indeed $\frac{dV}{dn} = Vr$

12 The expression $y = 500(1 - e^{-0.2t})$ represents the daily output of y units on day t of a production run. Find the instantaneous rate of change of the output y with respect to t.

$$y = 500 - 500 \times e^{-0.2t}$$

$$\frac{dy}{dt} = -500\, e^{-0.2t} \times (-0.2)$$

so $\frac{dy}{dt} = 100\, e^{-0.2t}$