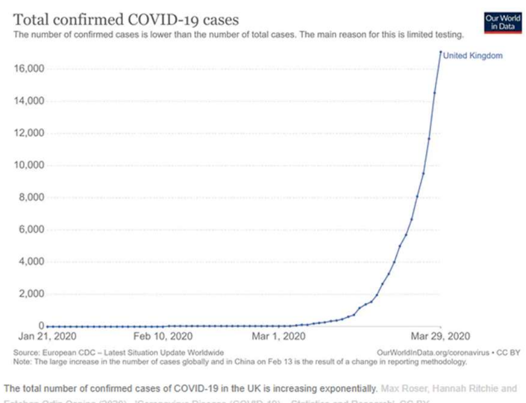
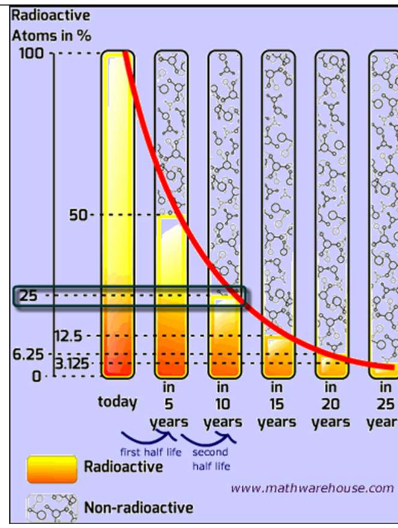
EXPONENTIAL GROWTH AND DECAY

A quantity has **exponential growth** when it increases by a constant percentage of its whole over a given period of time. This means that the larger the quantity at the start, the bigger the increase will be (e.g. if a country of 100 million people and a country of 10 million people are each growing their populations at an annual rate of 10%, then the first country's population increases by 10 million next year while the second country's population increases by only 1 million.

Exponential decay is when a quantity decreases by a constant percentage of its whole over a period of time.

Exponential growth	Exponential decay
$f(x) = A\, e^{kx}$ where k is a positive constant	$f(x) = A\, e^{-kx}$ where k is a positive constant





Differentiating $f(x) = A\, e^{kx}$ gives $\frac{df}{dx} = A\, k\, e^{kx}$ which simplifies as $\frac{df}{dx} = k \times f(x)$

$\frac{df}{dx}$ is called "**the exponential rate of change**".

EXPONENTIAL GROWTH AND DECAY

This means that exponential growth occurs when the rate of change of a quantity y with respect to another quantity x is proportional to y.

- the rate of growth of a colony of bacteria is proportional to the number of bacteria N present at any time, i.e. $\frac{dN}{dt} = kN$
- the rate of growth of a financial investment earning interest compounded at regular interval, i.e. $\frac{dI}{dt} = kI$
- the rate of decay of a radioactive isotope is proportional to the mass of that isotope present at any time, i.e. $\frac{dM}{dt} = -kM$ (the negative sign indicates decay, i.e. the amount reduces over time)
- the rate of cooling of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, i.e. $\frac{d\theta}{dt} = -k\theta$ where θ is the temperature difference at any time (this is Newton's law of cooling)
- the rate of decrease of atmospheric pressure with respect to height above sea level is proportional to the pressure at that height, i.e. $\frac{dP}{dt} = -kP$
- light passing through a transparent medium loses its intensity; the rate of loss of light intensity with respect to the distance is proportional to the light's intensity at the distance, i.e. $\frac{dI}{dx} = -kI$

Equations such as $\frac{df(x)}{dx} = k\, f(x)$ which contain a function and its derivative (i.e. $f(x)$ and $\frac{df(x)}{dx}$) are called **differential equations**.

In $y = A\, e^{kx}$, it is important to understand the role played by the constants A and k:

- at $x = 0$, $y = A\, e^{k\times 0} = A\, e^0 = A$ so if $x > 0$, then $\boldsymbol{A}$ is the initial value of $\boldsymbol{y}$
- because $\frac{dy}{dt} = ky$, $\boldsymbol{k}$ is the growth rate and influences the slope of the curve.

Example 4

The annual growth rate of the population of two towns P and Q are 10% and 5% respectively of their populations at any time. If the initial population of P is 20 000 and of Q is 10 000, find their populations 3 years later.

Solution

Let N be the population at any time t years.

For P: $\frac{dN}{dt} = 0.1N$

$\therefore N = Ae^{0.1t}$

At $t = 0$: $N = 20\,000$, so $A = 20\,000$

So: $N = 20\,000e^{0.1t}$

At $t = 3$: $N = 20\,000e^{0.3}$

$N \approx 27\,000$

For Q: $\frac{dN}{dt} = 0.05N$

$\therefore N = Ae^{0.05t}$

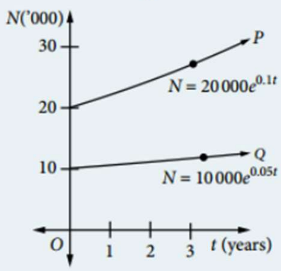
At $t = 0$: $N = 10\,000$, so $A = 10\,000$

So: $N = 10\,000e^{0.05t}$

At $t = 3$: $N = 10\,000e^{0.15}$

$N \approx 12\,000$

The answers are given to the nearest thousand.



The graphs indicate the roles played by A and k. Because A is the value of N when $t = 0$, it is the point where the curve crosses the N-axis.

The graph for P is steeper than the graph for Q. This indicates the role of k, the growth rate. The population of P grows more rapidly than the population of Q.

Example 5

A vessel containing water is being emptied. The volume $V(t)$ cubic metres of water remaining in the vessel after t minutes is given by $V(t) = Ae^{-kt}$.

(a) If $V(0) = 100$, find the value of A. **(b)** If $V(5) = 90$, find the value of k. **(c)** Find $V(20)$.

Solution

(a) When $t = 0$, $V = 100$, so: $A = 100$

(b) When $t = 5$, $V = 90$, so: $90 = 100e^{-5k}$

$0.9 = e^{-5k}$

$\therefore -5k = \log_e 0.9$

$k = -0.2\log_e 0.9 \approx 0.02$

(c) $V(t) \approx 100e^{-0.02t}$

$V(20) \approx 100e^{-0.4} \approx 67$ (Note that without rounding, this answer is closer to 66.)

Example 6

The pressure of the atmosphere, measured as P kilopascals (kPa), decreases with the altitude h km above sea level approximately according to $P = 101e^{-0.2h}$. Find the rate at which the air pressure falls with respect to height above sea level when: **(a)** $h = 5$ **(b)** $P = 20$

Solution

$$P = 101e^{-0.2h}$$

$$\frac{dP}{dh} = -0.2 \times 101e^{-0.2h}$$

$$\frac{dP}{dh} = -20.2e^{-0.2h} \quad [1]$$

or: $$\frac{dP}{dh} = -0.2P \quad [2]$$

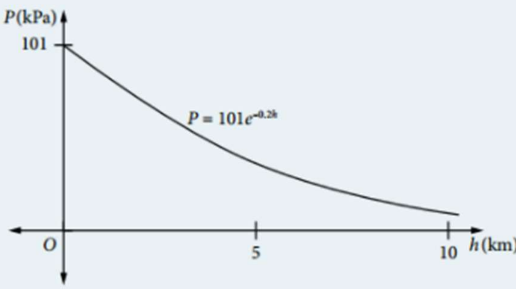
Here [1] gives $\frac{dP}{dh}$ as a function of h, while [2] gives $\frac{dP}{dh}$ as a function of P.

(a) $h = 5$: $\frac{dP}{dh} = -20.2e^{-1} = -7.43$

The pressure falls at a rate of 7.43 kPa/km when $h = 5$.

(b) $P = 20$: $\frac{dP}{dh} = -0.2 \times 20 = -4$

The pressure falls at a rate of 4 kPa/km when $P = 20$.



Example 7

The mass M of a radioactive substance is initially 10 grams. Twenty years later the mass of remaining radioactive substance is 9.6 grams.

(a) Find the annual decay rate, given that the rate of decay of a radioactive substance is proportional to the mass of the substance present at any time.

(b) In how many years will the mass of radioactive substance be halved?

Solution

It is known that $\frac{dM}{dt} = -kM$. You know that $M = Ae^{-kt}$ is a solution to this differential equation.

(a) $t = 0, M = 10$, so: $A = 10$

$t = 20, M = 9.6$: $9.6 = 10e^{-20k}$

$$\therefore -20k = \log_e 0.96$$

$$k = -\tfrac{1}{20}\log_e 0.96 \approx 0.002$$

(b) If $k = 0.002$: $M = 10e^{-0.002t}$

$M = 5$: $5 = 10e^{-0.002t}$

$$e^{-0.002t} = 0.5 \quad \text{or} \quad e^{0.002t} = 2$$

$$\therefore 0.002t = \log_e 2$$

$$t = 500\log_e 2 \approx 347$$

The mass of radioactive substance will be halved after about 347 years.

Note: the time taken for half of an amount of radioactive substance to decay is called the **half-life** of the substance.