THE GRADIENT AS A RATE OF CHANGE

1 A trench is being dug by a team of labourers who remove V cubic metres of soil in t minutes, where $V = 10t - \frac{t^2}{20}$.

(a) State the domain of the function, i.e. the values of t during which soil is being removed.
(b) At what rate is the soil being removed at the end of 40 minutes?
(c) Are the labourers working at a constant rate?
(d) What is their initial rate of work, i.e. when $t = 0$?
(e) At what time are they removing soil at the rate of 5 m^3 per minute?

a) $V(t)$ is a quadratic function, concave down, which can only take positive values. So $10t - \frac{t^2}{20} > 0 \Leftrightarrow t\left(10 - \frac{t}{20}\right) > 0$

$\Rightarrow t > 0$ or $10 - \frac{t}{20} > 0 \Rightarrow t < 200$. So $0 \leq t \leq 200$.

b) $\frac{dV}{dt} = 10 - \frac{2t}{20} = 10 - \frac{t}{10}$

So at $t = 40$ $\frac{dV}{dt} = 10 - \frac{40}{10} = 10 - 4 = 6\ m^3/min$

c) $\frac{dV}{dt}$ is not a constant as it depends of t, so they're not working at a constant rate.

d) $\frac{dV}{dt} = 10 - \frac{t}{10}$ so at $t = 0$ $\frac{dV}{dt} = 10 - \frac{0}{10} = 10\ m^3/min$

e) $\frac{dV}{dt} = 5$ when $10 - \frac{t}{10} = 5$

i.e. when $\frac{t}{10} = 5$ so at $t = 50$ min.

3 A cube of ice has an edge length of 10 cm. It melts so that its volume decreases at a constant rate and the block remains a cube. If the edge length measures 5 cm after 70 minutes, find:

(a) the rate at which the volume decreases **(b)** the volume at any time t.

$V = a^3$ with a the length of the edge.

a) $$\frac{dV}{dt} = \text{Constant} = \frac{\text{Initial Volume} - \text{Final Volume}}{\text{Difference in time}}$$

$$\frac{dV}{dt} = \frac{10^3 - 5^3}{0-70} = \frac{1000-125}{-70} = -12.5\,\text{cm}^3/\text{min}$$

b) $$\frac{dV}{dt} = -12.5\,\text{cm}^3/\text{min}$$

Note that the minus sign shows that the volume is decreasing.

It decreases linearly, so

$$V(t) = V(0) - 12.5 \times t$$

$$V(0) = 1000\,\text{cm}^3$$

so $$V(t) = 1000 - 12.5\,t$$

4 A water tank is being emptied. The quantity Q litres of water remaining in the tank at any time t minutes after it starts to empty is given by $Q(t) = 1000(20 - t)^2$, $t \geq 0$.

(a) At what rate is the tank being emptied at any time t?
(b) How much time does it take to empty the tank? (When is $V = 0$?)
(c) At what time is the water flowing out at a rate of 20 000 litres per minute?
(d) What is the average rate at which the water flows out in the first 5 minutes?

a) $Q(t) = 1000(400 - 40t + t^2)$

$Q(t) = 400{,}000 - 40{,}000t + 1000t^2$

$\frac{dQ}{dt} = -40{,}000 + 1{,}000 \times 2t = 2{,}000t - 40{,}000$

$\frac{dQ}{dt} = 2000(t - 20)$ L min^{-1}

b) $Q = 0$ when $1000(20-t)^2 = 0$, i.e when $t = 20$ min

(negative as it's decreasing)

c) $\frac{dQ}{dt} = -20{,}000$ when $2{,}000(t-20) = -20{,}000$

i.e. when $t - 20 = -10$, i.e when $t = 10$ min

d) $\frac{dQ}{dt} = 2{,}000t - 40{,}000$ so it's a linear function.

At $t = 0$ $\frac{dQ}{dt} = -40{,}000$

At $t = 5$ $\frac{dQ}{dt} = 2{,}000 \times 5 - 40{,}000 = -30{,}000$

So halfway between $t=0$ and $t = 5$ $\frac{dQ}{dt} = -35{,}000$

The average rate at which water flows out in the first 5 minutes is 35,000 L/min

6 A machine manufactures items at a variable rate given by $\frac{dQ}{dt} = 2t + 1, t \geq 0$, where Q is the number of items manufactured in a time t minutes.

At what rate is the machine working: **(a)** initially **(b)** after 10 minutes?

a) At $t=0$ $\quad \frac{dQ}{dt} = 2 \times 0 + 1 = 1$ item/min

b) At $t=10$ $\quad \frac{dQ}{dt} = 2 \times 10 + 1 = 21$ item/min

7 If the area of a circle is given by $A = \pi r^2$, show that the rate of change of the area with respect to the radius, $\frac{dA}{dr}$, is proportional to the radius. Find this rate when the radius is 2 cm.

$A(r) = \pi r^2$ so $\frac{dA}{dr} = \frac{d}{r}(\pi r^2) = \pi \frac{d r^2}{dr} = \pi \times 2r$

So indeed $\frac{dA}{dr}$ is proportional to r

When $r=2$, $\frac{dA}{dr} = 2\pi \times 2 = 4\pi$ cm/unit of time

8 A right circular cylinder of volume V has height h and radius of its base r. Find:

(a) the rate of change of volume with respect to height, if the radius of the base is constant

(b) the rate of change of volume with respect to the radius of the base, if the height is constant.

a) $V = \pi r^2 \times h$

if r is constant $\quad \frac{dV}{dh} = \frac{d}{dh}\left(\pi r^2 h\right) = \pi r^2 \frac{dh}{dh} = \pi r^2$

b) if the height h is constant,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\pi r^2 h\right) = \pi h \frac{dr^2}{dr} = \pi h \, 2r$$

So if h is constant $\quad \frac{dV}{dr} = 2\pi r h$

11 The revenue function for a particular manufacturer is $R = x\left(15 - \frac{x}{30}\right)$, where x is the number of units of the product sold. If the marginal revenue is given by $\frac{dR}{dx}$, find the marginal revenue when:

(a) $x = 6$ **(b)** $x = 15$ **(c)** $x = 225$

$$R = 15x - \frac{x^2}{30}$$

$$\text{so } \frac{dR}{dx} = 15 - \frac{2x}{30} = 15 - \frac{x}{15}$$

a) At $x = 6$ $\quad \frac{dR}{dx} = 15 - \frac{6}{15} = 14.6$

b) At $x = 15$ $\quad \frac{dR}{dx} = 15 - \frac{15}{15} = 15 - 1 = 14$

c) At $t = 225$ $\quad \frac{dR}{dx} = 15 - \frac{225}{15} = 0$