COMPOUND INTEREST APPLICATIONS

A practical example of a geometric sequence is the growth of money invested at compound interest. If \$100 is invested at the start of a year at compound interest of 8% p.a. (where 'p.a.' = 'per annum' = per year), its value is:

$\$100 \times 1.08$ at the end of the first year,

$\$100 \times (1.08)^2$ at the end of the second year,

$\$100 \times (1.08)^3$ at the end of the third year,

$\$100 \times (1.08)^n$ at the end of n years.

In general, if $\$P$ is invested at compound interest of r% p.a., it grows to:

PR at the end of the first year, where $R = 1 + \frac{r}{100}$,

PR^2 at the end of the second year,

PR^3 at the end of the third year,

PR^n at the end of n years.

These amounts form a geometric sequence whose common ratio is R. Thus you have the compound interest formula:

$$A_n = PR^n \quad \text{or} \quad A_n = P\left(1 + \frac{r}{100}\right)^n$$

Here P is the initial amount (or 'principal') and A_n is the amount that P grows to after n periods of time, where interest is applied at r% per period. The period is not necessarily one year:

18% p.a. = 9% per 6 months = 4.5% per 3 months = 1.5% per month, etc.

(For compound interest, this division of interest into shorter time periods is not precisely correct, but for simplicity you will calculate it as shown above.)

A_n is the nth term of the sequence. The corresponding series is investigated in the following examples.

Recurrence relation

This result for compound interest may also be written as a recurrence relation.

If $P = A_0$

then $A_1 = A_0 \times R$

and $A_2 = A_1 \times R$

and eventually $A_n = A_{n-1} \times R$

where A_n is the value of the initial amount after n time periods and

$R = 1 + \frac{r}{100}$, the interest rate being r% per time period.

Example 27

At the beginning of each year, Clementine contributes \$500 to her investment account. If her investment account earns compound interest paid at 8% p.a.,

(a) Calculate the value of the first contribution (\$500) at the end of 10 years.

(b) Draw the graph of $A_n = 500 \times 1.08^n$ for $0 \le n \le 10$.

(c) Calculate the accumulated value of her total investment (all contributions and interest) at the end of 10 years.

Solution

(a) Value of first contribution at end of first year $= 500 \times 1.08$

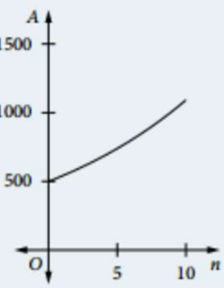
Value of first contribution at end of second year $= 500 \times 1.08^2$

Value of first contribution at end of tenth year $= 500 \times 1.08^{10}$

$= 1079.46$

The value of Clementine's first contribution at the end of 10 years is \$1079.46.

(b)



This graph shows that the amount is growing exponentially.

(c) Value of last contribution at end of tenth year $= 500 \times 1.08$ (1 year's interest)

Value of second-last contribution at end of tenth year $= 500 \times 1.08^2$ (2 years' interest)

Value of first contribution at end of tenth year $= 500 \times 1.08^{10}$ (10 years' interest)

The sum of all ten contributions $= 500 \times 1.08 + 500 \times 1.08^2 + 500 \times 1.08^3 + \ldots + 500 \times 1.08^{10}$

This is a finite geometric series of ten terms, with first term 500×1.08 and common ratio 1.08.

$$S_{10} = \frac{a\left(R^{10} - 1\right)}{R - 1}$$

$$= \frac{500 \times 1.08\left(1.08^{10} - 1\right)}{1.08 - 1}$$

$$= \frac{500 \times 1.08\left(1.08^{10} - 1\right)}{0.08}$$

$$= 7822.74$$

The total investment is worth \$7822.74 after 10 years.

In general, if someone invests \$$P$ at the beginning of each period of time at r% per period compound interest, the sum V of all investments at the end of n periods of time is given by:

$$V = PR + PR^2 + PR^3 + \ldots + PR^n$$

$$= PR(1 + R + R^2 + \ldots + R^{n-1})$$

$$= \frac{PR\left(R^n - 1\right)}{R - 1} \quad \text{where } R = 1 + \frac{r}{100}$$

Note that the *process* of obtaining this result is the important thing to remember, not the result.

Example 28

Your employer offers you two different salary packages: Package 1 offers a raise of $1300 each year for 5 years, while Package 2 offers a raise of $100 per month for 5 years.

(a) Which package will give you the largest salary after 5 years?

(b) Package 3 offers an annual increase of 4%. What further information will you need to determine the best of all three packages?

Solution

(a) Package 1: Starting salary = $\$P$, increase $1300 p.a. for 5 years

Salary after 5 years $= P + 5 \times 1300 = \$(P + 6500)$

Package 2: Starting salary = $\$P$, increase $100 p.m. for 60 months

Salary after 5 years $= P + 60 \times 100 = \$(P + 6000)$

Package 1 offers the largest salary after 5 years.

(b) To compare the first two salary packages to a percentage increase, you will need to know the starting salary.

Example 29

Amira takes out an interest-only loan for $400 000 at an interest rate of 6% p.a. compounded monthly. She pays back the loan amount plus interest after 6 months.

(a) Set up the recurrence relation for this loan where A_0 is the amount borrowed and A_n is the amount owing after n months (i.e. n lots of interest added).

(b) How much does she pay back after 6 months?

(c) How much interest does she pay after 6 months?

Solution

(a) $A_0 = 400\,000$, $r = 6\%$ p.a. $= \frac{6}{12}\%$ p.m. $= 0.5\%$ p.m.

$R = 1 + \frac{0.5}{100} = 1.005$

$A_1 = 400\,000 \times 1.005 = A_0 \times 1.005$

$A_2 = A_1 \times 1.005$

so $A_n = A_{n-1} \times 1.005$

(b) $n = 6$:

$A_6 = A_5 \times 1.005 = A_4 \times 1.005 \times 1.005 + A_4 \times 1.005^2$ etc.

This is obviously not an efficient way to find the amount.

Use $A_6 = A_0 \times 1.005^6$

$= 400\,000 \times 1.005^6$

$= 412\,151.00$

She pays back $412 151 after 6 months.

(c) Interest $= A_6 - A_0$

$= \$412\,151 - \$400\,000$

$= \$12\,151$

The terms **Future Value** and **Present Value** are frequently used in financial circles.

The Future Value of an investment is what it grows to, that is A_n that has been calculated in previous examples.

The Present Value of an investment is the amount that you need to start with to obtain a particular amount in the future. This is A_0 used in previous examples.

These terms were used when you looked at annuities at the beginning of the chapter.

Example 30

Georgia borrows \$5000 and agrees to make repayments of \$100 at the end of each month, calculated from the date of the loan. Interest is charged on the unpaid debt at 1.5% per month.

(a) How much does Georgia still owe after the eighth repayment?

(b) How much time will it take to pay off the loan?

(c) If Georgia doubles the repayments, how much time will it take to pay off the loan?

Solution

(a) $P = 5000$, $r = 0.015$, $R = 1.015$

Amount owing after first repayment $= 5000 \times 1.015 - 100$

Amount owing after second repayment $= (5000 \times 1.015 - 100) \times 1.015 - 100$

$= 5000 \times 1.015^2 - 100(1 + 1.015)$

Amount owing after third repayment $= (5000 \times 1.015^2 - 100(1 + 1.015)) \times 1.015 - 100$

$= 5000 \times 1.015^3 - 100(1 + 1.015 + 1.015^2)$

Amount owing after eighth repayment $= 5000 \times 1.015^8 - 100(1 + 1.015 + 1.015^2 + \ldots + 1.015^7)$

Now $1 + 1.015 + 1.015^2 + \ldots + 1.015^7$ is a geometric series with $a = 1$, $r = 1.015$, $n = 8$.

$\therefore$ Amount owing after eighth repayment $= 5000 \times 1.015^8 - \dfrac{100\left(1.015^8 - 1\right)}{1.015 - 1}$

$$= 5000 \times 1.015^8 - \frac{100\left(1.015^8 - 1\right)}{0.015}$$

$$= 4789.18$$

Thus Georgia still owes \$4789.18 after the eighth repayment.

(b) From part **(a)** you can see that after n repayments, the amount owing $= 5000 \times 1.015^n - \dfrac{100\left(1.015^n - 1\right)}{0.015}$

When the loan is paid off, the amount owing is 0, so you must solve $5000 \times 1.015^n - \dfrac{100\left(1.015^n - 1\right)}{0.015} = 0$

Solve: $0.015 \times 5000 \times 1.015^n - 100 \times 1.015^n + 100 = 0$

$$1.015^n(100 - 75) = 100$$

$$1.015^n = 4$$

$$n \log_{10} 1.015 = \log_{10} 4$$

$$n = \frac{\log_{10} 4}{\log_{10} 1.015} = 93.1$$

Georgia would take 94 months (7 years 10 months) to pay off the loan. The final payment is less than \$100.

(c) As for part **(b)**, but replacing the repayment value 100 with 200:

Solve: $0.015 \times 5000 \times 1.015^n - 200 \times 1.015^n + 200 = 0$

$$1.015^n(200 - 75) = 200$$

$$1.015^n = \frac{200}{125}$$

$$1.015^n = 1.6$$

$$n = \frac{\log_{10} 1.6}{\log_{10} 1.015} = 31.6$$

Georgia would take 32 months (2 years 8 months) to pay off the loan. Doubling the repayments reduces the time to pay off the loan by almost two-thirds.