OTHER USEFUL TECHNIQUES

1 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = \ldots$

A $2\int_0^{\frac{\pi}{2}} x\cos x\,dx$ $\boxed{\text{B}}$ 0 C $\pi-2$ D $\frac{\pi}{2}-1$

if $f(x) = x\cos x$, then $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$ so f odd

$\therefore \int_{-a}^{a} f(x)\,dx = 0 \quad \forall a \in \mathbb{R}.$

2 Evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{1+x}{1-x}}\,dx$ using: $\frac{1+x}{1-x} = \frac{(1+x)^2}{1-x^2}$

$$\int_0^{1/2} \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^{1/2} \sqrt{\frac{(1+x)^2}{(1-x^2)}}\,dx = \int_0^{1/2} \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$\text{——} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} + \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\text{——} = \left[\sin^{-1}x\right]_0^{1/2} - \frac{1}{2}\int_0^{1/2} \frac{(-2x)}{\sqrt{1-x^2}}\,dx$$

$$\text{——} = \left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right) - \frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^{1/2}$$

$$\text{——} = \frac{\pi}{6} - 0 - \left(\sqrt{1-\frac{1}{4}} - \sqrt{1}\right)$$

$$\text{——} = \frac{\pi}{6} - \left(\frac{\sqrt{3}}{2} - 1\right)$$

$$\text{——} = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$$

OTHER USEFUL TECHNIQUES

3 **(a)** Write $(x-1)(3-x)$ in the form $b^2-(x-a)^2$ where a and b are real numbers.

(b) Using the values of a and b from part **(a)** and making the substitution $x-a=b\sin\theta$, or otherwise, evaluate: $\int_1^3 \sqrt{(x-1)(3-x)}\,dx$

a) $$(x-1)(3-x) = -x^2+4x-3$$
$$= -(x^2-4x)-3$$
$$= -3-(x^2-4x)$$
$$= -3-\left[(x-2)^2-4\right]$$
$$= -3+4-(x-2)^2 = \underset{b=1}{1}^2-(x-\underset{a=2}{2})^2$$

b) $$I = \int_1^3 \sqrt{(x-1)(x-3)}\,dx = \int_1^3 \sqrt{1^2-(x-2)^2}\,dx.$$

Let $x-2=\sin\theta$ so $\dfrac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

When $x=1$ $\theta = -\pi/2$

When $x=3$ $\theta = \pi/2$

$$I = \int_{\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta} \times \cos\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta.$$

But $\cos 2\theta = 2\cos^2\theta - 1$ so $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$I = \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(\frac{-\pi}{2}+\frac{\sin(-\pi)}{2}\right)\right]$$

$$I = \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{2}$$

OTHER USEFUL TECHNIQUES

4 **(a)** Use the substitution $t = \tan\frac{x}{2}$ to find: $\int \sec x \, dx$

(b) By rewriting $\sec x$ as $\frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$, find: $\int \sec x \, dx$

a) $\sec x = \frac{1}{\cos x} = \frac{1+t^2}{1-t^2}$ with $t = \tan(x/2)$

$dx = \frac{2}{1+t^2} dt$

$$\int \sec x \, dx = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt$$

$$= -\ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$$

b) $$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.$$

We note that $\frac{d}{dx}(\tan x) = \sec^2 x$

and $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(-1)}{\cos^2 x} \times (-\sin x) = \tan x \times \sec x.$

$$\therefore \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

OTHER USEFUL TECHNIQUES

5 (a) Find: $\int \sec x \tan x \, dx$

(b) Using the substitution $u = \frac{\pi}{2} - x$, find: $\int \operatorname{cosec} x \cot x \, dx$

a) $$\int \sec x \tan x \, dx = \int \frac{\sin x}{\cos^2 x} dx = \int \frac{(-\cos x)'}{\cos^2 x} dx = -\int \frac{(\cos x)'}{\cos^2 x} dx$$

$$\therefore \int \sec x \tan x \, dx = \frac{1}{\cos x} + C = \sec x + C$$

b) $$\int \operatorname{cosec} x \cot x \, dx = \int \frac{1}{\sin x} \times \frac{\cos x}{\sin x} dx = \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \frac{(\sin x)'}{\sin^2 x} dx = -\frac{1}{\sin x} + C$$

$$\therefore \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

Alternatively let $u = \frac{\pi}{2} - x$, $\therefore \frac{du}{dx} = -1$ and $dx = -du$

$$x = \frac{\pi}{2} - u$$

$$\int \operatorname{cosec} x \cot x \, dx = \int \operatorname{cosec}\left(\frac{\pi}{2} - u\right) \cot\left(\frac{\pi}{2} - u\right) \times (-du)$$

$$= \int \frac{1}{\sin\left(\frac{\pi}{2} - u\right)} \times \frac{\cos(\pi/2 - u)}{\sin(\pi/2 - u)} \times (-du)$$

$$= \int \frac{1}{\cos u} \times \frac{\sin u}{\cos u} \times (-du)$$

$$= -\int \sec u \times \tan u \, du$$

$$= -\sec u + C = -\sec\left(\frac{\pi}{2} - x\right) + C$$

$$= \frac{-1}{\cos\left(\frac{\pi}{2} - x\right)} + C = -\frac{1}{\sin x} + C = -\operatorname{cosec} x + C$$

6 Find the length of the circumference of the circle $x^2 + y^2 = r^2$ using the arc length formula on the circle's first quadrant, i.e. arc length $= \int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

in the 1st quadrant $\quad y = \sqrt{r^2 - x^2} = (r^2 - x^2)^{1/2}$

So $\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2} \times (-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2}$ so $1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}} = \frac{r}{r\sqrt{1 - \left(\frac{x}{r}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{x}{r}\right)^2}}$

$\therefore I = \int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^r \frac{1}{\sqrt{1 - \left(\frac{x}{r}\right)^2}}\, dx$

let $u = \frac{x}{r}$, so $\frac{du}{dx} = \frac{1}{r}$ $\therefore$ $dx = r\, du$

$\therefore I = \int_0^1 \frac{r\, du}{\sqrt{1 - u^2}} = r\left[\sin^{-1} u\right]_0^1 = r\left[\sin^{-1} 1 - \sin^{-1} 0\right]$

$I = r\frac{\pi}{2}$ This is for the 1st quadrant only.

$\therefore$ for 4 quadrants, the length would be 4 times this amount, i.e $L = 4 \times r \times \frac{\pi}{2} = 2\pi r$

7 Let $I = \int_1^2 \frac{\cos^2\left(\frac{\pi}{6}x\right)}{x(3-x)}dx$.

(a) Use the substitution $u = 3 - x$ to show that: $I = \int_1^2 \frac{\sin^2\left(\frac{\pi}{6}u\right)}{u(3-u)}du$

(b) Hence find the value of I.

a) $u = 3 - x$, $x = 3 - u$, $\therefore \frac{du}{dx} = -1$ so $dx = -du$

when $x = 1$, $u = 2$ and when $x = 2$, $u = 1$,

$$I = \int_2^1 \frac{\cos^2\left[\frac{\pi}{6}(3-u)\right]}{(3-u) \times u} \times (-du) = \int_1^2 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi u}{6}\right)}{(3-u)\,u} du$$

$$I = \int_1^2 \frac{\sin^2\left(\frac{\pi u}{6}\right)}{u(3-u)} du \quad \text{or} \quad \int_1^2 \frac{\sin^2\left(\frac{\pi x}{6}\right)}{x(3-x)} dx$$

b) $$2I = \int_1^2 \frac{\cos^2\left(\frac{\pi x}{6}\right)}{x(3-x)} dx + \int_1^2 \frac{\sin^2\left(\frac{\pi x}{6}\right)}{x(3-x)} dx$$

$$2I = \int_1^2 \frac{1}{x(3-x)} dx$$

But $\frac{1}{x(3-x)} = \frac{a}{x} + \frac{b}{3-x} = \frac{(-a+b)x + 3a}{x(3-x)}$

so $a = 1/3$ and $-a + b = 0$ or $b = a = 1/3$.

$$\therefore 2I = \int_1^2 \left(\frac{1/3}{x} + \frac{1/3}{3-x}\right) dx = \frac{1}{3}\int_1^2 \left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

$$\therefore 2I = \frac{1}{3}\Big[\ln|x| - \ln|3-x|\Big]_1^2 = \frac{1}{3}\left[\ln\left|\frac{x}{3-x}\right|\right]_1^2$$

$$\therefore 2I = \frac{1}{3}\left[\ln\left|\frac{2}{1}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{1}{3}(\ln 2 + \ln 2) = \frac{2\ln 2}{3}$$

$$\therefore I = \frac{\ln 2}{3}$$

8 **(a)** Differentiate $x^2 \tan^{-1} x$ with respect to x. **(b)** Hence find: $\int 2x \tan^{-1} x \, dx$

a) $$\frac{d}{dx}\left(x^2 \tan^{-1} x\right) = 2x \tan^{-1} x + x^2 \times \frac{1}{1+x^2}$$

$$= 2x \tan^{-1} x + \frac{x^2}{1+x^2}$$

$$\therefore \; 2x \tan^{-1} x = \frac{d}{dx}\left(x^2 \tan^{-1} x\right) - \frac{x^2}{1+x^2}$$

b) $$\int 2x \tan^{-1} x \, dx = \int \left[\frac{d}{dx}\left(x^2 \tan^{-1} x\right) - \frac{x^2}{1+x^2}\right] dx$$

$$= x^2 \tan^{-1} x - \int \frac{x^2}{1+x^2} \, dx$$

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = 1 - \frac{1}{1+x^2} \quad \text{so}$$

$$\int 2x \tan^{-1} x \, dx = x^2 \tan^{-1} x + \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= x^2 \tan^{-1} x + x - \tan^{-1} x + C$$

$$\therefore \; \int 2x \tan^{-1} x \, dx = x^2 \tan^{-1} x - x - \tan^{-1} x + C$$

OTHER USEFUL TECHNIQUES

9 Show that: **(a)** $\int_{-1}^{1} x^3(1-x^2)^2\,dx = 0$ **(b)** $\int_{-1}^{1} x^2(1-x^2)^3\,dx = 2\int_0^1 x^2(1-x^2)^3\,dx$

a) if $f(x) = x^3(1-x^2)^2$ then $f(-x) = (-x)^3(1-(-x)^2)^2 = -f(x)$.

$\therefore$ f is an odd function.

$\therefore$ $\forall a \in \mathbb{R}$ $\int_{-a}^{a} f(x)\,dx = 0$

$\therefore$ particularly for $a = 1$, $\int_{-1}^{1} x^3(1-x^2)^2\,dx = 0$

b) if $g(x) = x^2(1-x^2)^3$

then $g(-x) = (-x)^2\left[1-(-x)^2\right]^3 = g(x)$.

$\therefore$ g is an even function.

$\therefore$ $\forall a \in \mathbb{R}$ $\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx$

particularly for $a = 1$ $\int_{-1}^{1} x^2(1-x^2)^3\,dx = 2\int_0^1 x^2(1-x^2)^3\,dx$

OTHER USEFUL TECHNIQUES

10 **(a)** By using an appropriate substitution, show that: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(b) Hence show that: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\,dx$

a) let $u = a + b - x$ or $x = a + b - u$.

when $x = a$, $u = a + b - a = b$

when $x = b$, $u = a + b - b = a$

$\frac{dx}{du} = -1$ or $du = -dx$

$\therefore \int_a^b f(x)\,dx = \int_b^a f(a+b-u) \times (-du) = -\int_b^a f(a+b-u)\,du$

$\therefore \int_a^b f(x)\,dx = \int_a^b f(a+b-u)\,du = \int_a^b f(a+b-x)\,dx$.

b) $\int_{\pi/6}^{\pi/3} \sin x\,dx = \int_{\pi/6}^{\pi/3} \sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right) dx$

$\qquad = \int_{\pi/6}^{\pi/3} \sin\left(\frac{\pi}{2} - x\right) dx$

$\qquad = \int_{\pi/6}^{\pi/3} \cos x\,dx$

12 **(a)** If $x > 0$ and $1 < u < 1 + x$, show that: $\frac{1}{1+x} < \frac{1}{u} < 1$

(b) By integrating each term of $\frac{1}{1+x} < \frac{1}{u} < 1$ with respect to u between 1 and $(1 + x)$, show that: $\frac{x}{1+x} < \log_e(1 + x) < x$

a) $1 < u < 1 + x \Rightarrow 1 > \frac{1}{u} > \frac{1}{1+x}$

or $\frac{1}{1+x} < \frac{1}{u} < 1$

b) $\therefore \int_1^{1+x} \frac{1}{1+x}\, du < \int_1^{1+x} \frac{1}{u}\, du < \int_1^{1+x} 1 \times du$

$\Rightarrow \frac{1}{1+x} \int_1^{1+x} du < \int_1^{1+x} \frac{du}{u} < \int_1^{1+x} du$

$\Leftrightarrow \frac{1}{1+x} \left[u\right]_1^{1+x} < \left[\ln|u|\right]_1^{1+x} < \left[u\right]_1^{1+x}$

$\Leftrightarrow \frac{1}{1+x}(1+x-1) < \ln|1+x| - \ln 1 < (1+x-1)$

$\Leftrightarrow \frac{x}{1+x} < \ln(1+x) < x$