3 Evaluate the following:

(a) $\int_{-1}^{2} 3x(2-x)dx$ **(b)** $4\int_{-3}^{-1} x(x+1)^2 dx$ **(c)** $\int_{-2}^{4} (x^3-2)dx$

a) $\int_{-1}^{2} 3x(2-x)\,dx = \int_{-1}^{2}(6x-3x^2)\,dx = \left[\frac{6x^2}{2} - \frac{3x^3}{3}\right]_{-1}^{2} = \left[3x^2 - x^3\right]_{-1}^{2}$

$= \left[3\times 2^2 - 2^3\right] - \left[3\times(-1)^2 - (-1)^3\right] = (12-8)-(3+1) = 0$

b) $4\int_{-3}^{-1} x(x+1)^2\,dx = 4\int_{-3}^{-1} x(x^2+2x+1)\,dx = 4\int_{-3}^{-1}(x^3+2x^2+x)\,dx$

$= 4\left[\frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2}\right]_{-3}^{-1}$

$= 4\left[\left(\frac{(-1)^4}{4} + \frac{2\times(-1)^3}{3} + \frac{(-1)^2}{2}\right) - \left(\frac{(-3)^4}{4} + \frac{2(-3)^3}{3} + \frac{(-3)^2}{2}\right)\right]$

$= 4\left(\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) - \left(\frac{81}{4} - \frac{54}{3} + \frac{9}{2}\right)\right) = 4\left[\frac{1}{12} - \frac{27}{4}\right] = -\frac{80}{3} = -26\frac{2}{3}$

c) $\int_{-2}^{4}(x^3-2)\,dx = \left[\frac{x^4}{4} - 2x\right]_{-2}^{4}$

$= \left(\frac{4^4}{4} - 2\times 4\right) - \left(\frac{(-2)^4}{4} - 2\times(-2)\right)$

$= (64-8) - (4+4)$

$= 48$

7 Evaluate: **(a)** $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(\sin 2x - \cos 2x)\,dx$ **(b)** $\int_{0}^{\frac{\pi}{2}}(\cos 2x - x)\,dx$ **(c)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sec^2 x}{\tan x}\,dx$

a) $\displaystyle\int_{\pi/8}^{\pi/4}(\sin 2x - \cos 2x)\,dx = \left[-\frac{\cos 2x}{2} - \frac{\sin 2x}{2}\right]_{\pi/8}^{\pi/4}$

$\displaystyle = \left[-\frac{\cos(\pi/2)}{2} - \frac{\sin(\pi/2)}{2}\right] - \left[-\frac{\cos(\pi/4)}{2} - \frac{\sin(\pi/4)}{2}\right]$

$\displaystyle = -\frac{1}{2} - \left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right) = -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}$

b) $\displaystyle\int_{0}^{\pi/2}(\cos 2x - x)\,dx = \left[\frac{\sin 2x}{2} - \frac{x^2}{2}\right]_{0}^{\pi/2}$

$\displaystyle = \left[\frac{\sin\pi}{2} - \frac{(\pi/2)^2}{2}\right] - \left[\frac{\sin 0}{2} - 0\right]$

$\displaystyle = -\frac{\pi^2}{8}$

c) $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\sec^2 x}{\tan x}\,dx = \Big[\ln(\tan x)\Big]_{\pi/6}^{\pi/3} = \left[\ln\left(\tan\frac{\pi}{3}\right)\right] - \left[\ln\left(\tan\frac{\pi}{6}\right)\right]$

$\displaystyle \tan\frac{\pi}{3} = \frac{\sin\pi/3}{\cos\pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\displaystyle \tan\frac{\pi}{6} = \frac{\sin\pi/6}{\cos\pi/6} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

So $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\sec^2 x}{\tan x}\,dx = \ln(\sqrt{3}) - \ln\left(\frac{1}{\sqrt{3}}\right)$

$= \ln(\sqrt{3}) + \ln(\sqrt{3})$

$= 2\ln(\sqrt{3})$

$\displaystyle = 2\ln 3^{1/2} = 2\times\frac{1}{2}\ln 3 = \ln 3$

17 Evaluate: **(a)** $\int_{-2}^{2}\left(e^{x}-e^{-x}\right)dx$ **(b)** $\int_{-1}^{2}\left(e^{x}-e^{-x}\right)^{2}dx$ **(c)** $\int_{1}^{3}\left(e^{x}+\frac{1}{x}\right)dx$

a) $\int_{-2}^{2}(e^{x}-e^{-x})\,dx = \left[e^{x}+e^{-x}\right]_{-2}^{2} = (e^{2}+e^{-2})-(e^{-2}+e^{-(-2)})$

$= e^{2}+e^{-2}-e^{-2}-e^{2} = 0$

b) $\int_{-1}^{2}(e^{x}-e^{-x})^{2}dx = \int_{-1}^{2}(e^{2x}-2+e^{-2x})\,dx$

$= \left[\frac{e^{2x}}{2}-2x+\frac{e^{-2x}}{(-2)}\right]_{-1}^{2}$

$= \left(\frac{e^{4}}{2}-2\times 2+\frac{e^{-4}}{(-2)}\right)-\left(\frac{e^{-2}}{2}-2\times(-1)+\frac{e^{2}}{(-2)}\right)$

$= \frac{e^{4}-e^{-4}}{2}-\frac{e^{-2}-e^{2}}{2}-4-2 \simeq 24.92$

c) $\int_{1}^{3}\left(e^{x}+\frac{1}{x}\right)dx = \left[e^{x}+\ln x\right]_{1}^{3}$

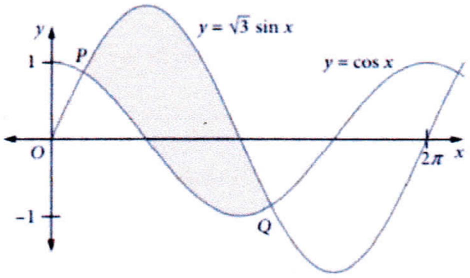
$= (e^{3}+\ln 3)-(e+\ln 1)$

$= e^{3}+\ln 3-e$

21 The diagram shows the graphs of $y = \sqrt{3}\sin x$ and $y = \cos x$. The first two points of intersection to the right of the y-axis are labelled P and Q.

(a) Solve the equation $\sqrt{3}\sin x = \cos x$ to find the abscissae of P and Q.

(b) Find the area of the shaded region in the diagram.



a) $\sqrt{3}\sin x = \cos x \iff \dfrac{\sin x}{\cos x} = \dfrac{1}{\sqrt{3}} \iff \tan x = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

so $x = \pi/6$ or $x = \dfrac{\pi}{6} + \pi = \dfrac{7\pi}{6}$

b) This area is $\displaystyle\int_{\pi/6}^{7\pi/6} (\sqrt{3}\sin x - \cos x)\,dx$.

$$\int_{\pi/6}^{7\pi/6} (\sqrt{3}\sin x - \cos x)\,dx = \Big[\sqrt{3}(-\cos x) - \sin x\Big]_{\pi/6}^{7\pi/6}$$

$$= \left[-\sqrt{3}\cos\left(\frac{7\pi}{6}\right) - \sin\left(\frac{7\pi}{6}\right)\right] - \left[-\sqrt{3}\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\right]$$

$$= \left[-\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right)\right] - \left[-\sqrt{3} \times \frac{\sqrt{3}}{2} - \frac{1}{2}\right]$$

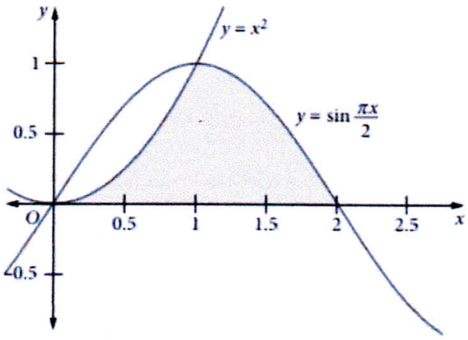
$$= \left[\frac{3}{2} + \frac{1}{2}\right] - \left[\frac{-3}{2} - \frac{1}{2}\right]$$

$$= 2 - (-2) = 4$$

23 The shaded region in the diagram is bounded by the curves $y = \sin\frac{\pi x}{2}$, $y = x^2$ and the x-axis.

(a) Show that the two curves meet at $x = 1$.

(b) Calculate the exact area of the shaded region.



a) at $x = 1$ $\quad x^2 = 1$

and $\sin\left(\frac{\pi \times 1}{2}\right) = 1$ too

So the two curves meet at $x = 1$

b) this area is $\int_0^1 x^2\,dx + \int_1^2 \sin\left(\frac{\pi x}{2}\right) dx = A$

$$A = \left[\frac{x^3}{3}\right]_0^1 + \left[-\cos\left(\frac{\pi x}{2}\right) \times \frac{2}{\pi}\right]_1^2$$

$$A = \frac{1}{3} - \frac{2}{\pi}\left[\cos\left(\frac{\pi x}{2}\right)\right]_1^2$$

$$A = \frac{1}{3} - \frac{2}{\pi}\left[\cos\left(\frac{\pi \times 2}{2}\right) - \cos\left(\frac{\pi}{2} \times 1\right)\right]$$

$$A = \frac{1}{3} - \frac{2}{\pi}\left(\cos\pi - \cos\frac{\pi}{2}\right)$$

$$A = \frac{1}{3} - \frac{2}{\pi}(-1) = \frac{1}{3} + \frac{2}{\pi}$$

$$A = \frac{\pi + 6}{3\pi}$$

26 The velocity v m s^{-1} of a particle moving in a straight line is given by $v = 6t^2 - 4t + 1$ ($t \geq 0$). The particle initially has a displacement -10 m from O. Find:

(a) the displacement and acceleration at any time t

(b) the acceleration when the velocity is 3 m s^{-1}

(c) the velocity when the acceleration is 20 m s^{-2}.

a) $v(t) = 6t^2 - 4t + 1$ so $a(t) = 12t - 4$

and $x(t) = 6\frac{t^3}{3} - 4\frac{t^2}{2} + t + C = 2t^3 - 2t^2 + t + C$

At $t = 0$ $x(0) = -10$ so $C = -10$

$$x(t) = 2t^3 - 2t^2 + t - 10$$

b) $v(t) = 3 \Leftrightarrow 6t^2 - 4t + 1 = 3 \Leftrightarrow 6t^2 - 4t - 2 = 0$

$\Delta = 16 - 4 \times (-2) \times 6 = 64 = (8)^2 =$

So $t = \frac{4+8}{2 \times 6} = \frac{12}{12} = 1$. (the other value of t is negative)

For this value of t, $a(1) = 12 \times 1 - 4 = 8$ m s^{-2}

c) $a(t) = 20$ when $12t - 4 = 20$ or $12t = 24$ $t = 2$

For this value of t, $v(2) = 6 \times 2^2 - 4 \times 2 + 1$

$$v(2) = 17 \text{ m s}^{-1}$$

27 A full water tank holds 4000 litres. When the tap is turned on, water flows out from the tank at a rate of $\frac{dV}{dt} = 110 + 17t - t^2$ litres per minute, where t is the time in minutes since the tap was turned on.

(a) At what time is the tank emptying at a rate of 50 litres per minute?
(b) Find the volume of water that has flowed out of the since the tap was turned on as a function of t.
(c) How much water has flowed out of the tank 12 minutes after the tap was turned on?
(d) When does the water stop flowing out of the tank?
(e) How much water is left in the tank (to the nearest litre) when the water stops flowing out of the tank?

a) $\frac{dV}{dt} = 110 + 17t - t^2$ $\quad \frac{dV}{dt} = 50$ when $110 + 17t - t^2 = 50$

$\Leftrightarrow -t^2 + 17t + 60 = 0 \quad \Delta = 17^2 - 4 \times 60 \times (-1) = 529 = 23^2$

$t = \frac{-17 - 23}{2(-1)} = 20$ min (the other value is negative, ∴ impossible).

b) $\frac{dV}{dt} = 110 + 17t - t^2$ So $V(t) = 110t + 17\frac{t^2}{2} - \frac{t^3}{3} + C$

where V is the volume of water that has flowed out of the tank.

So $\underbrace{\left[-V(t)\right]}_{v(t)} = -110t - 17\frac{t^2}{2} + \frac{t^3}{3} - C$ is the volume of water in the tank. At $t = 0$ $(-V(t)) = 4000$ so $C = -4000$

$v(t) = -110t - 17 \times \frac{t^2}{2} + \frac{t^3}{3} + 4{,}000$

c) at $t = 12$ $\quad v(t) = -110 \times 12 - 17 \times \frac{12^2}{2} + \frac{12^3}{3} + 4000 = 2032$

As there was originally 4,000 litres in the tank, that means 1,968 L have flowed out

d) $v(t) = 0$ when $\frac{dV}{dt} = 0$ or $110 + 17t - t^2 = 0$

$\Delta = 17^2 - 4 \times 110 \times (-1) = 729 = 27^2 \quad t = \frac{-17 - 27}{-2} = 22$ s

the other value is negative ∴ impossible

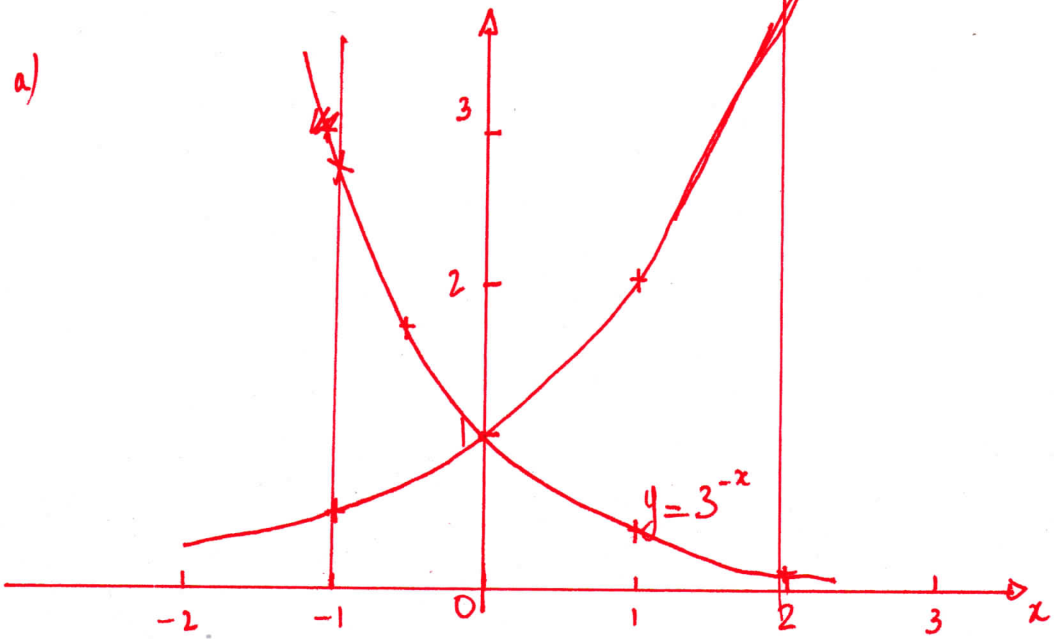
e) At $t = 22$ s, $\quad v(t) = -110 \times 22 - 17 \times \frac{22^2}{2} + \frac{22^3}{3} + 4{,}000$

$v(22) = 1{,}015$ Litres.

29 **(a)** Sketch the graphs of $y = 2^x$ and $y = 3^{-x}$ over the domain $-1 \le x \le 2$.

(b) Calculate the area of the region bounded by the curves $y = 2^x$, $y = 3^{-x}$ and the ordinates $x = -1$ and $x = 2$.

a)



b) between -1 and 0, $y = 3^{-x}$ is above $y = 2^x$

—— 0 and 2, $y = 3^{-x}$ is below ——.

So this area is $\displaystyle\int_{-1}^{0} (3^{-x} - 2^x)\,dx + \int_0^2 (2^x - 3^{-x})\,dx = \text{Area}$

$$\text{Area} = \int_{-1}^{0} \left(e^{-x\ln 3} - e^{x\ln 2}\right) dx + \int_0^2 \left(e^{x\ln 2} - e^{-x\ln 3}\right) dx$$

$$\text{Area} = \left[\frac{e^{-x\ln 3}}{(-\ln 3)} - \frac{e^{x\ln 2}}{\ln 2}\right]_{-1}^{0} + \left[\frac{e^{x\ln 2}}{\ln 2} - \frac{e^{-x\ln 3}}{(-\ln 3)}\right]_0^2$$

$$\text{Area} = \left[\frac{-3^{-x}}{\ln 3} - \frac{2^x}{\ln 2}\right]_{-1}^{0} + \left[\frac{2^x}{\ln 2} + \frac{3^{-x}}{\ln 3}\right]_0^2$$

$$\text{Area} = \left[\left(\frac{-1}{\ln 3} - \frac{1}{\ln 2}\right) - \left(\frac{-3}{\ln 3} - \frac{1}{2\ln 2}\right)\right] + \left[\left(\frac{2^2}{\ln 2} + \frac{1}{9\ln 3}\right) - \left(\frac{1}{\ln 2} + \frac{1}{\ln 3}\right)\right]$$

$$\text{Area} = \frac{2}{\ln 3} - \frac{1}{2\ln 2} + \frac{3}{\ln 2} - \frac{8}{9\ln 3} = \frac{10}{9\ln 3} + \frac{5}{2\ln 2} \simeq 4.62 \text{ units}^2$$