APPLICATIONS OF CALCULUS - CHAPTER REVIEW

1 By integration, find the volume of the solid of revolution formed from the region bounded by:

(a) the circle $x^2 + y^2 = 1$, rotated about the x-axis

(b) the line $y = x + 3$ between $x = 0$ and $x = 2$, rotated about the x-axis

(c) the parabola $y = x^2 + 3$ between $y = 4$, $y = 12$ and the y-axis, rotated about the y-axis.

a) $x^2 + y^2 = 1$ so $y = \sqrt{1 - x^2}$

this volume is $\pi \int_{-1}^{1} y^2 \, dx = \pi \int_{-1}^{1} (1 - x^2) \, dx = \pi \left[x - \frac{x^3}{3} \right]_{-1}^{1}$

$V = \pi \left[\left(1 - \frac{1^3}{3}\right) - \left((-1) - \frac{(-1)^3}{3}\right) \right] = \pi \left[\frac{2}{3} - \left(\frac{-2}{3}\right) \right] = \frac{4\pi}{3}$

b) This volume is $\pi \int_{0}^{2} (x+3)^2 \, dx = \pi \int_{0}^{2} (x^2 + 6x + 9) \, dx$

$V = \pi \left[\frac{x^3}{3} + \frac{6x^2}{2} + 9x \right]_0^2 = \pi \left[\frac{x^3}{3} + 3x^2 + 9x \right]_0^2$

$V = \pi \left[\frac{2^3}{3} + 3 \times 4 + 18 \right] = \frac{98\pi}{3}$

c) This volume is $\pi \int_{4}^{12} x^2 \, dy = \pi \int_{4}^{12} (y - 3) \, dy$

$V = \pi \left[\frac{y^2}{2} - 3y \right]_4^{12} = \pi \left[\left(\frac{12^2}{2} - 3 \times 12 \right) - \left(\frac{4^2}{2} - 3 \times 4 \right) \right]$

$V = \pi \left[36 - (8 - 12) \right]$

$V = 40\pi$

3 Using the substitution $u=\sqrt{x}$, find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.

$u = \sqrt{x}$ so $\frac{du}{dx} = \frac{1}{2}x^{1/2 - 1} = \frac{1}{2\sqrt{x}}$ so $\frac{dx}{\sqrt{x}} = 2\,du$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^u \times 2\,du = 2\int e^u\,du = 2e^u + C$

$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2e^{\sqrt{x}} + C$

4 **(a)** Use the substitution $u = 1-x$ to evaluate $\int_0^1 2x\sqrt{1-x}\,dx$.

a) $u = 1-x$ $\quad \frac{du}{dx} = -1$ so $dx = -du$

when $x=0$ $u=1$; when $x=1$, $u=0$

$I = \int_1^0 2(1-u)\sqrt{u} \times (-du) = \int_0^1 2(1-u)\sqrt{u}\,du$

$I = 2\left[\int_0^1 \sqrt{u}\,du - \int_0^1 u\sqrt{u}\,du\right] = 2\left[\left[\frac{u^{3/2}}{3/2}\right]_0^1 - \left[\frac{u^{5/2}}{5/2}\right]_0^1\right]$

$I = 2\left[\frac{2}{3}\times 1 - \frac{2}{5}\times 1\right] = \frac{8}{15}$

5 Using the substitution $u = x^3 + 1$, or otherwise, evaluate $\int_0^1 x^2 e^{x^3+1}\,dx$.

$$\frac{du}{dx} = 3x^2 \quad \therefore \quad x^2\,dx = \frac{du}{3}$$

when $x = 0$, $u = 1$; when $x = 1$, $u = 2$

$$I = \int_1^2 e^u \times \frac{du}{3} = \frac{1}{3}\int_1^2 e^u\,du = \frac{1}{3}\left[e^u\right]_1^2$$

$$I = \frac{1}{3}\left[e^2 - e\right] = \frac{e}{3}(e-1)$$

$\boxed{\uparrow}$ 6 Evaluate $\int_0^{\frac{\pi}{4}} \sin\theta\cos^2\theta\,d\theta$. $= \int_0^{\pi/4} (-\cos\theta)'\cos^2\theta\,d\theta = -\int_0^{\pi/4} (\cos\theta)'\cos^2\theta\,d\theta$

$$I = -\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/4} = -\frac{1}{3}\left[\cos^3\theta\right]_0^{\pi/4} = \frac{1}{3}\left[\cos^3\theta\right]_{\pi/4}^{0}$$

$$I = \frac{1}{3}\left[1 - \left(\frac{\sqrt{2}}{2}\right)^3\right] = \frac{1}{3}\left[1 - \frac{2\sqrt{2}}{8}\right]$$

$$I = \frac{8 - 2\sqrt{2}}{24} = \frac{4 - \sqrt{2}}{12}$$

7 Use the substitution $u = \log_e x$ to evaluate $\int_e^{e^3} \frac{1}{x(\log_e x)^2} dx$.

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{dx}{x} = du$

when $x = e$, $u = 1$; when $x = e^3$, $u = 3$

$I = \int_1^3 \frac{du}{u^2} = \left[\frac{u^{-2+1}}{-2+1}\right]_1^3 = \left[\frac{u^{-1}}{-1}\right]_1^3 = \left[-\frac{1}{u}\right]_1^3 = \left[\frac{1}{u}\right]_3^1$

$I = \frac{1}{1} - \frac{1}{3} = \frac{2}{3}$

P 11 (a) Differentiate $e^{2x}(2\sin x - \cos x)$. (b) Hence, or otherwise, find $\int e^{2x} \sin x \, dx$.

a) $f(x) = e^{2x}(2\sin x - \cos x)$ Product rule

$f'(x) = 2e^{2x}(2\sin x - \cos x) + e^{2x}(2\cos x + \sin x)$

$f'(x) = e^{2x}[5\sin x]$

b) $\int e^{2x} \sin x \, dx = \left[\int \frac{d}{dx}\left[e^{2x}(2\sin x - \cos x)\right] dx\right] \times \frac{1}{5}$

$= \frac{1}{5}\left(e^{2x}(2\sin x - \cos x)\right) + C$

$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$

14 **(a)** By expanding the left-hand side, show that $\sin(6x + 3x) + \sin(6x - 3x) = 2\sin 6x \cos 3x$.

(b) Hence find $\int \sin 6x \cos 3x \, dx$.

a) $\sin(6x + 3x) + \sin(6x - 3x) = \sin 6x \cos 3x + \cos 6x \sin 3x$

$+ \sin 6x \cos(-3x) + \cos 6x \sin(-3x)$

$= \sin 6x \cos 3x + \cancel{\cos 6x \sin 3x} + \sin 6x \cos 3x$

$- \cancel{\cos 6x \sin 3x}$

$= 2 \sin 6x \cos 3x$

b) $\int \sin 6x \cos 3x \, dx = \frac{1}{2}\int \left[\sin 9x + \sin 3x\right] dx$

$= \frac{1}{2}\left[-\frac{\cos 9x}{9} - \frac{\cos 3x}{3}\right] + C$

$= -\frac{\cos 9x}{18} - \frac{\cos 3x}{6} + C$

15 Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the x-axis and the line $x = \frac{\pi}{6}$ is rotated about the x-axis.

This is $V = \pi \int_0^{\pi/6} \sin^2 2x \, dx$

But $\cos 2\theta = 1 - 2\sin^2\theta \quad \therefore \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$V = \pi \int_0^{\pi/6} \frac{1 - \cos 4x}{2} \, dx = \frac{\pi}{2} \int_0^{\pi/6} (1 - \cos 4x) \, dx$$

$$V = \frac{\pi}{2} \left[x - \frac{\sin 4x}{4} \right]_0^{\pi/6}$$

$$V = \frac{\pi}{2} \left[\frac{\pi}{6} - \frac{\sin\left(\frac{4\pi}{6}\right)}{4} \right] = \frac{\pi^2}{12} - \frac{\pi}{8} \sin\left(\frac{2\pi}{3}\right)$$

$$V = \frac{\pi^2}{12} - \frac{\pi}{8} \times \frac{\sqrt{3}}{2}$$

$$V = \frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{16} = \frac{16\pi^2 - 12\pi\sqrt{3}}{192}$$

$$V = \frac{4\pi^2 - 3\pi\sqrt{3}}{48}$$