COUNTING TECHNIQUES IN PROBABILITY

This section builds on the counting techniques developed earlier in the chapter. A brief summary of the necessary knowledge follows.

Factorial notation

$n! = n(n-1)(n-2)\ldots\times 3\times 2\times 1 \qquad 0! = 1$

Permutations

A permutation is an *ordered* selection or arrangement of all or part of a set of objects.

- The number of arrangements of n different objects is $n!$
- The number of arrangements of r objects selected from n different objects is ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$
- The number of ways of arranging n objects when there are p identical objects of one kind, q identical objects of another kind, etc. is $\dfrac{n!}{p!q!\ldots}$

Combinations

A combination is an *unordered* selection of all or part of a set of objects.
A combination (or 'selection') is an unordered permutation (or 'arrangement').

- The number of combinations of r objects selected from n different objects is ${}^{n}C_{r} = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Useful results

$${}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and} \quad {}^{n}C_{n-r} = \binom{n}{n-r} = \frac{n!}{(n-r)!r!}$$

Thus: ${}^{n}C_{r} = {}^{n}C_{n-r}$ **or** $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

Probability

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}, \quad 0 \le P(A) \le 1 \qquad P(\bar{A}) = 1 - P(A)$$

Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$
Not mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

Example 23

PINs (Personal Identification Numbers) are short number codes used as a security device.

(a) How many four-digit PINs can be made using the digits 0, 1, 2, … 9?
(b) What is the probability that someone can randomly guess the four-digit PIN?
(c) As a further security measure, some financial institutions require a four-symbol code that is made using the 26 letters of the alphabet and/or the digits 0–9. How many different four-symbol codes can be created?
(d) What is the probability of someone correctly guessing one of these four-symbol codes?
(e) What is the probability of someone correctly guessing the four-digit PIN and the correct four-symbol code?

Solution

(a) Number of PINs $= 10^4 = 10\,000$ **(b)** $P(\text{guessing PIN}) = \dfrac{1}{10^4} = \dfrac{1}{10\,000}$
(c) There are 26 letters of the alphabet and 10 digits. Number of codes $= 36^4 = 1\,679\,616$
(d) $P(\text{guessing symbol code}) = \dfrac{1}{36^4} = \dfrac{1}{1\,679\,616}$
(e) $P(\text{guessing both PIN and symbol code}) = \dfrac{1}{10^4} \times \dfrac{1}{36^4} = \dfrac{1}{16\,796\,160\,000}$

Example 25

Eight people in total, including two people A and B, randomly arrange themselves in a straight line. What is the probability that:

(a) A and B are next to each other

(b) A and B are not next to each other

(c) A and B occupy the end positions

(d) there are at least three people between A and B?

Solution

(a) Eight people can be arranged in a row in 8! ways = 40 320
As A and B are next to each other, consider them as one unit.
This unit and the other six people can be arranged in 7! ways.
A and B can be arranged in two ways (AB or BA).
Total ways of arranging the people $= 2 \times 7!$

$$P(A \text{ and } B \text{ next to each other}) = \frac{2 \times 7!}{8!} = \frac{2 \times 7!}{8 \times 7!} = \frac{1}{4}$$

(b) $P(A \text{ and } B \text{ not next to each other}) = 1 - P(A \text{ and } B \text{ next to each other}) = 1 - \frac{1}{4} = \frac{3}{4}$

(c) A and B can be placed at the ends in two ways (A at the beginning and B at the end, or B at the beginning and A at the end). The remaining six places can be filled in 6! ways.
Total number of ways with A and B at the ends $= 2 \times 6!$

$$P(A \text{ and } B \text{ occupy end positions}) = \frac{2 \times 6!}{8!} = \frac{2 \times 6!}{8 \times 7 \times 6!} = \frac{1}{28}$$

(d) A diagram helps to visualise the possibilities:

A _ _ _ B _ _ _, A _ _ _ _ B _ _, A _ _ _ _ _ B _, A _ _ _ _ _ _ B.
_ A _ _ _ B _ _, _ A _ _ _ _ B _, _ A _ _ _ _ _ B.
_ _ A _ _ _ B _, _ _ A _ _ _ _ B.
_ _ _ A _ _ _ B.

A and B can be swapped, so this gives $10 \times 2 = 20$ arrangements for A and B with 6! ways of filling the remaining six places.

Total number of favourable arrangements $= 20 \times 6!$

$$P(\text{at least three people between } A \text{ and } B) = \frac{20 \times 6!}{8!} = \frac{20 \times 6!}{8 \times 7 \times 6!} = \frac{5}{14}$$

Example 26

A bag contains nine cubes: three that are white and six that are black. Two cubes are drawn at random without replacement. Calculate the probability that both cubes are black.

Solution

Method 1

Number of ways two cubes can be selected from nine cubes without replacement $= {}^9C_2$
Number of ways the two cubes can be black $= {}^6C_2$

$$P(\text{both cubes are black}) = \frac{{}^6C_2}{{}^9C_2} = \frac{6 \times 5}{2 \times 1} \times \frac{2 \times 1}{9 \times 8} = \frac{5}{12}$$

Method 2

Let A be the event 'black cube in the first draw' and B be the event 'black cube in the second draw'.
Each cube is equally likely to be selected, so $P(A) = \frac{6}{9}$.

The outcome of the second draw is dependent on the outcome of the first draw:
if event A has happened then there are eight cubes left, of which five are black.

Hence $P(B|A) = \frac{5}{8}$

Now $P(AB) = P(A) \times P(B|A) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$

Example 27

A group of nine people contains three teachers and six students. A random sample of five people is selected. What is the probability that it contains: **(a)** exactly two teachers **(b)** not more than two teachers?

Solution

(a) Exactly two teachers means the sample contains two teachers and three students.

Number of ways of selecting five people from nine people $=\binom{9}{5}$

Number of ways of selecting two teachers from three teachers $=\binom{3}{2}$

Number of ways of selecting three students from six students $=\binom{6}{3}$

Number of ways of selecting two teachers and three students $=\binom{3}{2}\times\binom{6}{3}$

Hence: $P(\text{two teachers and three students})=\dfrac{\binom{3}{2}\times\binom{6}{3}}{\binom{9}{5}}$

Remember $\binom{9}{5}=\binom{9}{4}$: $P(\text{two teachers and three students})=\dfrac{\binom{3}{2}\times\binom{6}{3}}{\binom{9}{5}}=3\times\dfrac{6\times5\times4}{3\times2\times1}\times\dfrac{4\times3\times2\times1}{9\times8\times7\times6}=\dfrac{10}{21}$

(b) Not more than two teachers means either zero teachers and five students, one teacher and four students or two teachers and three students.

Zero teachers and five students can be selected in $\binom{6}{5}$ ways.

One teacher and four students can be selected in $\binom{3}{1}\times\binom{6}{4}$ ways.

Two teachers and three students can be selected in $\binom{3}{2}\times\binom{6}{3}$ ways.

Hence: $P(\text{not more than two teachers})=\dfrac{\binom{6}{5}+\binom{3}{1}\times\binom{6}{4}+\binom{3}{2}\times\binom{6}{3}}{\binom{9}{5}}=\dfrac{6+45+60}{126}=\dfrac{37}{42}$

Example 29

A box contains 10 pairs of headphones, two of which are defective. A sample of three pairs of headphones is drawn at random from the box without replacement. Find the probability that not more than one pair of headphones is defective.

Solution

Number of possible selections = ${}^{10}C_3 = 120$

Number of favourable selections = [0 defective] + [1 defective]

'0 defective' means that 0 (none) of the two defectives are selected and three of the eight non-defectives are selected,

i.e. ${}^{2}C_0 \times {}^{8}C_3$ ways.

'1 defective' means that one of the two defectives is selected and two of the eight non-defectives are selected,

i.e. ${}^{2}C_1 \times {}^{8}C_2$ ways.

Number of favourable selections = ${}^{2}C_0 \times {}^{8}C_3 + {}^{2}C_1 \times {}^{8}C_2 = \frac{8\times7\times6}{3\times2\times1} + 2\times\frac{8\times7}{2\times1} = 56 + 56 = 112$

$P(\text{not more than 1 defective}) = \frac{112}{120} = \frac{14}{15}$

Example 30

A scientific study of penguins uses the 'capture–recapture' technique. In the first stage of the study, 36 penguins are caught, tagged and then released. Later, in the second stage of the study, some penguins are again captured from the same area. Of these penguins, 27 of them are found to be tagged, which is 30% of the total number captured in this second stage.

(a) In the second stage of the study, how many penguins are captured in total?

(b) Calculate the estimate for the total population of penguins in this area.

Solution

(a) 27 is 30% of total captured, N: $27 = 0.3N$

$N = \frac{27}{0.3} = 90$ Hence 90 penguins were captured.

(b) P is the total population of penguins in the area.

Use the ratio: $\frac{27}{90} = \frac{36}{P}$

$P = \frac{36\times90}{27} = 120$

or: $0.3 = \frac{36}{P}$ The penguin population is about 120.