OTHER REPRESENTATIONS OF COMPLEX NUMBERS

1 Write each complex number in both polar and Cartesian form.

(a) $e^{\frac{i\pi}{3}}$ **(b)** $e^{\frac{i\pi}{2}}$ **(c)** $e^{\frac{5\pi i}{6}}$ **(d)** $e^{\frac{i\pi}{4}}$ **(e)** $e^{\frac{-i\pi}{2}}$ **(f)** $e^{\frac{-2\pi i}{3}}$ **(g)** $e^{1-\frac{i\pi}{2}}$ **(h)** $e^{2+\frac{i\pi}{3}}$

a) $e^{i\pi/3} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ b) $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i \times 1 = i$

c) $e^{\frac{i5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i \times \frac{1}{2}$ d) $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

e) $e^{-i\pi/2} = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = 0 + i \times (-1) = -i$

f) $e^{-\frac{2i\pi}{3}} = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} + i \times \left(-\frac{\sqrt{3}}{2}\right)$

g) $e^{1-i\pi/2} = e^1 \times e^{-\frac{i\pi}{2}} = e \times \left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right] = e[0 + i \times (-1)] = -ie$

h) $e^{2+\frac{i\pi}{3}} = e^2 e^{i\pi/3} = e^2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$\quad = e^2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

$\quad = \frac{e^2}{2} + i\frac{e^2\sqrt{3}}{2}$

OTHER REPRESENTATIONS OF COMPLEX NUMBERS

2 Write each complex number in the form $re^{i\theta}$, giving any decimal answers correct to two decimal places, where necessary.

(a) $3(\cos 1.5 + i\sin 1.5)$ (b) $-\sqrt{3}+i$ (c) $3+2i$ (d) $4(\cos 2 - i\sin 2)$

(e) $2-2i$ (f) $4\left(-\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ (g) $-2-2\sqrt{3}i$ (h) $\left(1+\sqrt{2}\right)+\left(1-\sqrt{2}\right)i$

a) $3(\cos 1.5 + i\sin 1.5) = 3e^{i\times 1.5}$

b) $-\sqrt{3}+i = 2\left[-\frac{\sqrt{3}}{2}+\frac{i}{2}\right] = 2e^{i5\pi/6}$ $\quad r=\sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$

c) $3+2i = \sqrt{13}\left[\frac{3}{\sqrt{13}} + i\frac{2}{\sqrt{13}}\right] = \sqrt{13}\,e^{i\theta}$ $\quad r=\sqrt{3^2+2^2} = \sqrt{13}$

$\theta \simeq 0.59$ (rad)

d) $4(\cos 2 - i\sin 2) = 4(\cos(2) + i\sin(-2)) = 4e^{-2i}$

e) $2-2i = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] = 2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = 2\sqrt{2}\,e^{-i\pi/4}$ $\quad r=\sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

f) $4\left[-\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right] = 4\left[\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right] = 4e^{i4\pi/5}$

g) $-2-2\sqrt{3}i = 4\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 4\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

$= 4e^{-\frac{i2\pi}{3}}$

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h) $(1+\sqrt{2}) + (1-\sqrt{2}i)$ $\quad r=\sqrt{(1+\sqrt{2})^2+(1-\sqrt{2})^2}$

$= \sqrt{6}\left[\frac{1+\sqrt{2}}{\sqrt{6}} + i\left(\frac{1-\sqrt{2}}{\sqrt{6}}\right)\right]$ $\quad r=\sqrt{1+2+\cancel{2\sqrt{2}}+1+2-\cancel{2\sqrt{2}}}$

$r=\sqrt{6}$

$\cos\theta = \frac{1+\sqrt{2}}{\sqrt{6}}$ and $\sin\theta = \frac{1-\sqrt{2}}{\sqrt{6}}$ so $\theta \simeq 0.17$ (radians)

$(1+\sqrt{2}) + i(1-\sqrt{2}) = \sqrt{6}\,e^{-i\times 0.17}$

3 If $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$, then the complex number $\frac{1}{2}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)$ is equal to:

A $e^{\frac{5\pi i}{8}}$ **B** $e^{\frac{-5\pi i}{8}}$ **C** $e^{\frac{i\pi}{8}}$ **D** $e^{\frac{-i\pi}{8}}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ so $\cos^2\frac{\pi}{8} = \left(\frac{\sqrt{2}}{2}+1\right)\times\frac{1}{2} = \frac{1}{2}\left(\frac{\sqrt{2}+2}{2}\right)$ so $\cos\frac{\pi}{8} = \frac{\sqrt{\sqrt{2}+2}}{2}$

and $\sin^2\frac{\pi}{8} = 1 - \cos^2\frac{\pi}{8} = 1 - \frac{1}{2}\left(\frac{\sqrt{2}+2}{2}\right) = \frac{4-\sqrt{2}-2}{4} = \frac{2-\sqrt{2}}{4}$

so $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$

So $\frac{1}{2}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right) = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$ Response $\boxed{C}$

4 (a) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, write an expression for $e^{-i\theta}$.

(b) Using part **(a)**, obtain expressions for $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

a) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ ①

b) We know that $e^{i\theta} = \cos\theta + i\sin\theta$ ②

Adding equation ① and ②, we obtain by elimination:

$2\cos\theta = e^{i\theta} + e^{-i\theta}$ so $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

NOTE that the function $f(x) = \frac{e^{x}+e^{-x}}{2}$ is called "hyperbolic cosine" abbreviation "cosh"

For sine: $2i\sin\theta = e^{i\theta} - e^{-i\theta}$

so $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

NOTE that the function $f(x) = \frac{e^{\cancel{ix}}-e^{-ix}}{2}$ is called "hyperbolic sine" abbreviation "sinh"

OTHER REPRESENTATIONS OF COMPLEX NUMBERS

6 (a) Write $z = 1 - \sqrt{3}i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 (ii) z^3 (iii) z^5 (iv) $\sqrt{z}$ (v) $\frac{1}{z}$

a) $z = 1 - \sqrt{3}i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right] = 2e^{-i\pi/3}$

i) $z^2 = \left[2e^{-i\pi/3}\right]^2 = 4e^{-\frac{2i\pi}{3}} = 4\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right] = 4\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

ii) $z^3 = \left[2e^{-i\pi/3}\right]^3 = 8e^{-i\pi} = 8(-1) = -8$

iii) $z^5 = z^3 \times z^2 = (-8) \times 4e^{-2i\pi/3} = -32e^{-2i\pi/3} = 32e^{i\pi/3}$

$z^5 = +32\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = 32\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 16 + 16\sqrt{3}i$

iv) $\sqrt{z} = z^{1/2} = \left[2e^{-i\pi/3}\right]^{1/2} = \sqrt{2}e^{\frac{-i\pi}{6}} = \sqrt{2}\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]$

$\sqrt{z} = \sqrt{2}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

v) $\frac{1}{z} = \frac{1}{2e^{-i\pi/3}} = \frac{1}{2}e^{i\pi/3}$

or $\frac{1}{z} = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$\frac{1}{z} = \frac{1}{2}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = \frac{1}{4} + i\frac{\sqrt{3}}{4}$

9 (a) Given $e^{i\theta} = \cos\theta + i\sin\theta$, write expressions for $e^{3i\theta}$ and $\left(e^{i\theta}\right)^3$.

(b) Hence write expressions for $\cos 3\theta$ in terms of $\cos\theta$ and $\cos^3\theta$.

(c) Hence write expressions for $\sin 3\theta$ in terms $\sin\theta$ and $\sin^3\theta$.

a) $\left(e^{i\theta}\right)^3 = \left[\cos\theta + i\sin\theta\right]^3 = (\cos\theta + i\sin\theta)(\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)$

$= \cos^3\theta + 2i\sin\theta\cos^2\theta - \cos\theta\sin^2\theta + i\sin\theta\cos^2\theta + 2i^2\cos\theta\sin^2\theta - i\sin^3\theta$

$= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[-\sin^3\theta + 3\sin\theta\cos^2\theta\right]$

But $\left(e^{i\theta}\right)^3 = e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

b) By equalling real and imaginary parts for both expressions, we obtain: $\underline{\cos 3\theta} = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$
$= 4\cos^3\theta - 3\cos\theta$

c) $\sin 3\theta = -\sin^3\theta + 3\sin\theta(1-\sin^2\theta) = -4\sin^3\theta + 3\sin\theta$

10 Given that $\dfrac{1-3i}{1+2i} = re^{i\theta}$, $r > 0$ and for $-\pi < \theta \le \pi$, find the values of r and θ.

$\dfrac{1-3i}{1+2i} = \dfrac{(1-3i)(1-2i)}{(1+2i)(1-2i)} = \dfrac{1-2i-3i+6i^2}{1-4i^2} = \dfrac{-5-5i}{5} = -(1+i)$

$= -\sqrt{2}\left[\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right] = \sqrt{2}\left[-\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right]$

$= \sqrt{2}\left[\cos\left(\dfrac{-3\pi}{4}\right) + i\sin\left(\dfrac{-3\pi}{4}\right)\right]$

$= \sqrt{2}\, e^{-i3\pi/4}$

14 Given $z_1 = 2e^{\frac{i\pi}{8}}$, $z_2 = 3e^{\frac{5i\pi}{12}}$, $z_3 = \frac{1}{3}e^{\frac{-5i\pi}{6}}$ and $z_4 = \frac{1}{2}e^{\frac{-3i\pi}{4}}$, find the polar form for each of the following, plotting each one on the Argand plane.

(a) $z_1^2 \times z_4$ **(b)** $z_2 \times z_3$ **(c)** $z_1 \times z_2 \times z_3 \times z_4$ **(d)** $\frac{z_1^2}{z_4}$ **(e)** $\frac{\sqrt{z_3}}{z_2}$

a) $z_1^2 \times z_4 = \left(2e^{i\pi/8}\right)^2 \times \frac{1}{2} e^{-i\frac{3\pi}{4}} = 2e^{i\left(\frac{\pi}{4} - \frac{3\pi}{4}\right)} = 2e^{-i\pi/2} = -2i$

b) $z_2 \times z_3 = 3e^{i5\pi/12} \times \frac{1}{3} e^{-i5\pi/6} = e^{i\left[\frac{5\pi}{12} - \frac{5\pi}{6}\right]} = e^{-i\frac{5\pi}{12}}$

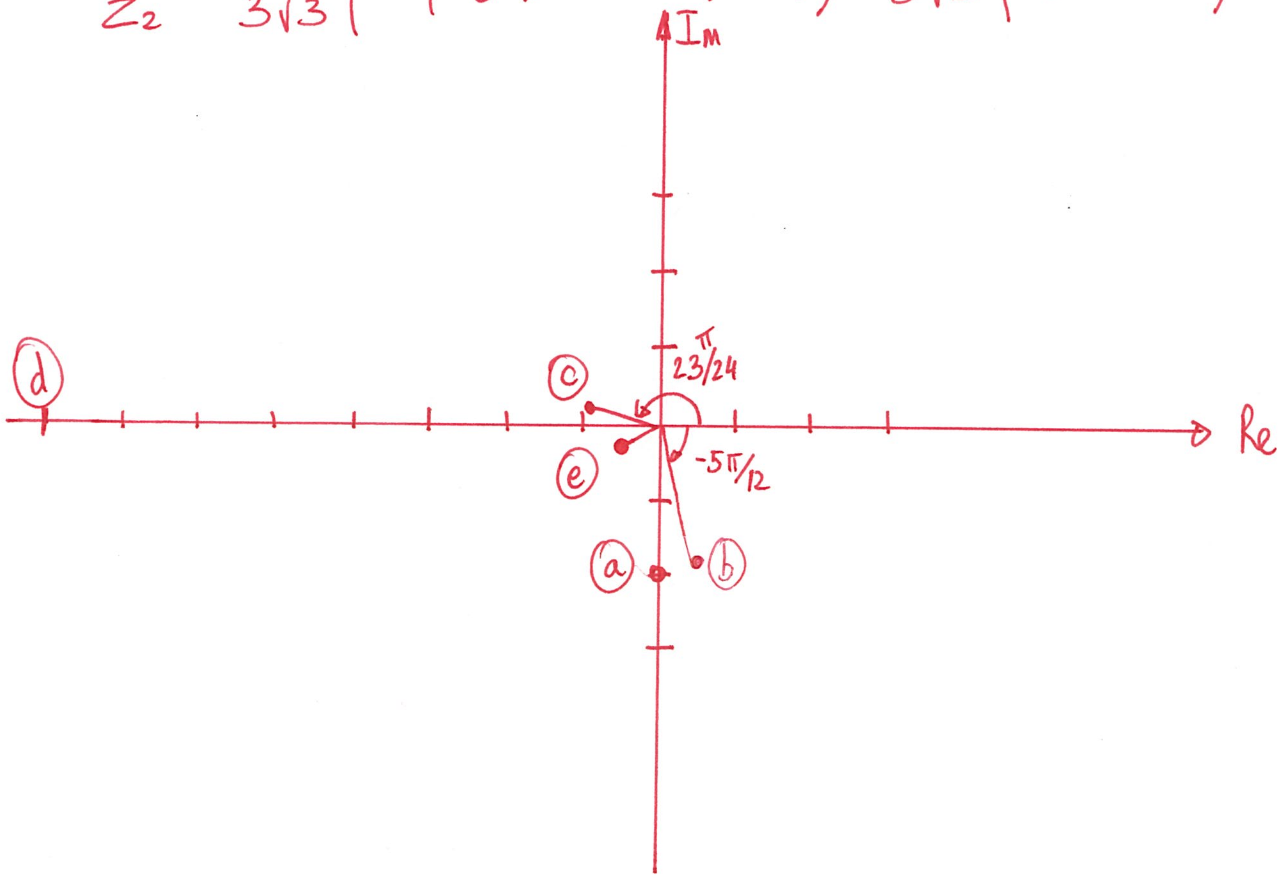
c) $z_1 \times z_2 \times z_3 \times z_4 = 2e^{\frac{i\pi}{8}} \times e^{-\frac{i5\pi}{12}} \times \frac{1}{2} e^{-\frac{3i\pi}{4}} = e^{i\left(\frac{\pi}{8} - \frac{5\pi}{12} - \frac{3\pi}{4}\right)} =$

$= e^{-i\frac{25}{24}\pi} = e^{i\frac{23}{24}\pi}$ (as the argument has to be between $-\pi$ and π)

d) $\frac{z_1^2}{z_4} = \frac{\left(2e^{i\pi/8}\right)^2}{\frac{1}{2}e^{-\frac{3\pi i}{4}}} = 8e^{i\left(\pi/4 + \frac{3\pi}{4}\right)} = 8e^{i\pi} = -8$

e) $\frac{\sqrt{z_3}}{z_2} = \frac{\left(\frac{1}{3}e^{-5i\pi/6}\right)^{1/2}}{3e^{5i\pi/12}} = \frac{1}{3\sqrt{3}} e^{i\left(-\frac{5\pi}{12} - \frac{5\pi}{12}\right)} = \frac{1}{3\sqrt{3}} e^{i\left(-\frac{5\pi}{6}\right)}$

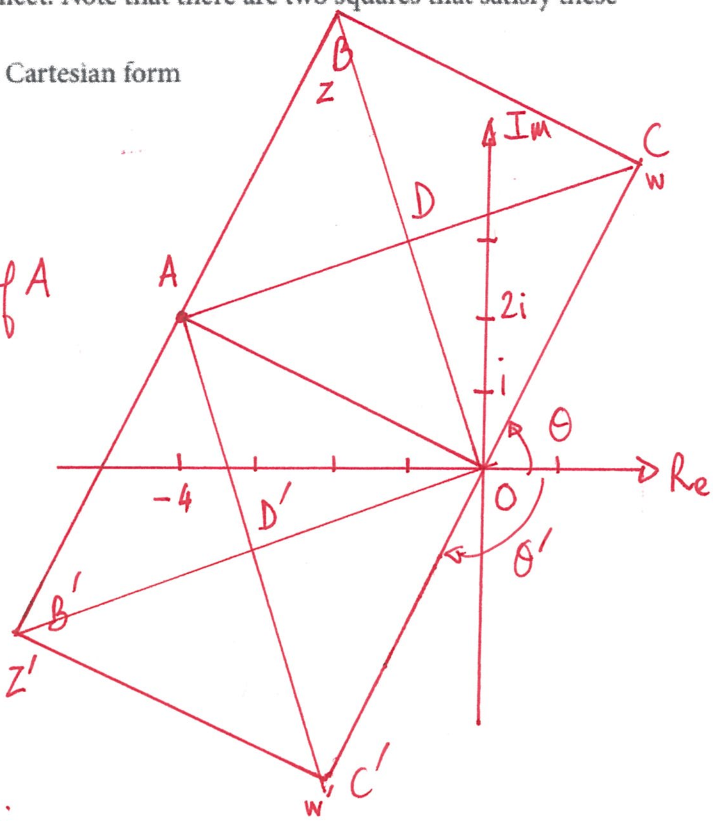
$\frac{\sqrt{z_3}}{z_2} = \frac{1}{3\sqrt{3}}\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(\frac{-5\pi}{6}\right)\right) = \frac{1}{3\sqrt{3}}\left(-\frac{\sqrt{3}}{2} - i \times \frac{1}{2}\right) = -\frac{1}{2\sqrt{3}} - \frac{i}{6\sqrt{3}}$



16 *OABC* is a square on an Argand diagram. *O* represents 0, *A* represents $-4+2i$, *B* represents z, *C* represents w and *D* is the point where the diagonals of the square meet. Note that there are two squares that satisfy these requirements. For each square, find:

(a) the complex numbers represented by *C* and *D* in Cartesian form

(b) the value of $\arg\left(\frac{w}{z}\right)$.



a) To find the coordinates of C, we multiply by $(-i)$ the coordinates of A

So $Z_C = (-i)\times(-4+2i) = 2+4i$

And then the middle of AC is

$$\frac{Z_A+Z_C}{2} = \frac{(-4+2i)+(2+4i)}{2}$$

$$= \frac{-2+6i}{2} = -1+3i$$

For the other square:

$$Z_{C'} = i\times(-4+2i) = -2-4i$$

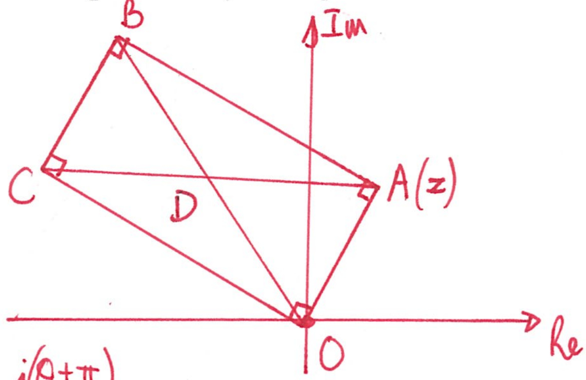
$$\frac{Z_A+Z_{C'}}{2} = \frac{-4+2i-2-4i}{2} = -3-i$$

b) For C $\quad \arg\left(\frac{w}{z}\right) = \arg(w) - \arg(z) = \theta - \left(\theta+\frac{\pi}{4}\right) = -\frac{\pi}{4}$

For C' $\quad \arg\left(\frac{w'}{z'}\right) = \arg(w') - \arg(z') = \theta' - \left(\theta'-\frac{\pi}{4}\right)$

so $\arg\left(\frac{w'}{z'}\right) = -\left(\frac{-\pi}{4}\right) = \frac{\pi}{4}$

17 On an Argand diagram, *OABC* is a rectangle. The length of *OC* is twice the length of *OA*. The vertex *A* corresponds to the complex number *z*. Find the complex number represented by *D*, the point of intersection of the diagonals *OB* and *AC*.



$Z_A = re^{i\theta}$

To obtain Z_C, we rotate Z_A of $\frac{\pi}{2}$ and times by 2, so

$$Z_C = 2re^{i\theta} \times i = 2re^{i(\theta+\frac{\pi}{2})}$$

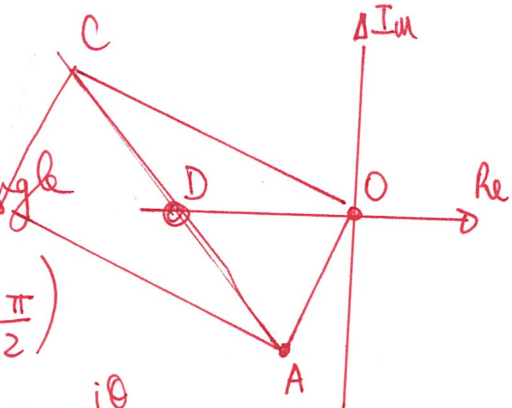
Then D is the middle of AC, so

$$Z_D = \frac{re^{i\theta} + 2ri\,e^{i\theta}}{2}$$

$$Z_D = re^{i\theta}\left[\frac{1}{2} + i\right] = Z_A\left[\frac{1}{2} + i\right] = Z_A\left[r'e^{i\theta'}\right]$$

$$r' = \sqrt{\left(\frac{1}{2}\right)^2 + 1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$Z_D = Z_A \times \frac{\sqrt{5}}{2}\left[\frac{1}{\sqrt{5}} + \frac{2i}{\sqrt{5}}\right]$$



OR considering the other possible rectangle

To get ZC we rotate Z_A by $\frac{3\pi}{2}$ $\left(\text{or } -\frac{\pi}{2}\right)$

So $Z_C = re^{i\theta} \times 2 \times (i^3) = -2ri\,e^{i\theta}$

then $Z_D = \dfrac{Z_A + Z_C}{2} = \dfrac{re^{i\theta} - 2rie^{i\theta}}{2} = re^{i\theta}\left(\dfrac{1}{2} - i\right)$

$$Z_D = Z_A\left(\frac{1}{2} - i\right)$$