# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**1** Find: **(a)** $\int \sin^2 x\,dx$ **(b)** $\int \cos^2 2x\,dx$ **(c)** $\int \sin^2 \frac{x}{2}\,dx$ **(d)** $\int \cos^2 3x\,dx$

a) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ so $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left[x - \int \cos 2x\,dx\right] = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$

So $\displaystyle\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

b) $\cos 4x = \cos^2 2x - \sin^2 2x = 2\cos^2 2x - 1$ so $\cos^2 2x = \dfrac{1+\cos 4x}{2}$

$$\int \cos^2 2x\,dx = \int \frac{1+\cos 4x}{2}\,dx = \frac{1}{2}\left[x + \int \cos 4x\,dx\right] + C$$

So $\displaystyle\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{2}\,\frac{\sin 4x}{4} + C = \frac{x}{2} + \frac{\sin 4x}{8} + C$

c) $\cos x = 1 - 2\sin^2 \frac{x}{2}$ so $\sin^2 \frac{x}{2} = \dfrac{1-\cos x}{2}$

$$\int \sin^2 \frac{x}{2}\,dx = \int \frac{1-\cos x}{2}\,dx = \frac{1}{2}\left[x - \int \cos x\,dx\right] + C$$

So $\displaystyle\int \sin^2 \frac{x}{2}\,dx = \frac{x}{2} - \frac{1}{2}\sin x + C = \frac{x - \sin x}{2} + C$

d) $\cos 6x = \cos^2 3x - \sin^2 3x = 2\cos^2 3x - 1$

so $\cos^2 3x = \dfrac{1+\cos 6x}{2}$

$$\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{1}{2}\left[x + \int \cos 6x\,dx\right] + C$$

$$\text{———} = \frac{1}{2}\left[x + \frac{\sin 6x}{6}\right] + C$$

$$\text{———} = \frac{x}{2} + \frac{\sin 6x}{12} + C$$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**3** Find: **(a)** $\int \sin^2 x \cos x \, dx$ **(b)** $\int \tan x \sec^2 x \, dx$ **(c)** $\int \cos^3 x \sin x \, dx$

a) $\int \sin^2 x \cos x \, dx = \int \sin^2 x \, (\sin x)' \, dx = \frac{\sin^3 x}{3} + C$

b) $\int \tan x \sec^2 x \, dx = \int \tan x \, (\tan x)' \, dx = \frac{\tan^2 x}{2} + C$

c) $\int \cos^3 x \sin x \, dx = \int \cos^3 x \, (-\cos x)' \, dx$

$\underline{\qquad\qquad} = -\int \cos^3 x \, (\cos x)' \, dx$

$\underline{\qquad\qquad} = -\frac{\cos^4 x}{4} + C$

<u>Alternative</u> let $u(x) = \cos x$ so $\frac{du}{dx} = -\sin x$

so $du = -\sin x \times dx$

$\sin x \, dx = -du$

$\int \cos^3 x \sin x \, dx = \int u^3 \times (-du)$

$\underline{\qquad\qquad} = -\int u^3 \, du = -\frac{u^4}{4} + C$

$\underline{\qquad\qquad} = -\frac{\cos^4 x}{4} + C$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**3** Find: **(g)** $\int \sin x \cos^4 x\, dx$ **(h)** $\int \sec^2 x \sin x\, dx$ **(i)** $\int \operatorname{cosec}^2 x \cos x\, dx$

g) $\int \sin x \cos^4 x\, dx = \int \cos^4 x\, (-\cos x)'\, dx$

$= -\int \cos^4 x\, (\cos x)'\, dx$

$= -\dfrac{\cos^5 x}{5} + C$

h) $\int \sec^2 x \sin x\, dx = \int \dfrac{1}{\cos^2 x} \sin x\, dx = \int (\cos x)^{-2} (-\cos x)'\, dx$

$= -\dfrac{(\cos x)^{-2+1}}{(-2+1)} + C = -\dfrac{(\cos x)^{-1}}{(-1)} + C$

$= \dfrac{1}{\cos x} + C = \sec x + C$

i) $\int \operatorname{cosec}^2 x \cos x\, dx = \int \dfrac{1}{\sin^2 x} \cos x\, dx = \int (\sin x)^{-2} (\sin x)'\, dx$

$= \dfrac{(\sin x)^{-2+1}}{-2+1} + C$

$= \dfrac{(\sin x)^{-1}}{(-1)} + C$

$= -\dfrac{1}{\sin x} + C$

$= -\operatorname{cosec} x + C$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**3** Find: (g) $\int 2\cos^2\frac{x}{2}\,dx$ (h) $\int \sin^2\left(\frac{\pi}{2}-x\right)dx$ (i) $\int \sin x\cos 2x\,dx$

g) $\int 2\cos^2\frac{x}{2}\,dx = \int [\cos x + 1]\,dx$ as $\cos x = 2\cos^2\frac{x}{2} - 1$

(from $\cos 2x = 2\cos^2 x - 1$)

$\_\_\_ = \sin x + x + C$

h) $\int \sin^2\left(\frac{\pi}{2}-x\right)dx = \int \cos^2 x\,dx = \int \left(\frac{\cos 2x + 1}{2}\right)dx$

$\_\_\_ = \frac{1}{2}\int (\cos 2x + 1)\,dx = \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right] + C$

$\_\_\_ = \frac{\sin 2x}{4} + \frac{x}{2} + C$

i) $\int \sin x \cos 2x\,dx = \int \frac{1}{2}\left[\sin(x+2x) + \sin(x-2x)\right]dx$

(From $\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$)

So $\int \sin x\cos 2x\,dx = \frac{1}{2}\int\left[\sin 3x + \sin(-x)\right]dx$

$\_\_\_ = \frac{1}{2}\int\left[\sin 3x - \sin x\right]dx$

$\_\_\_ = \frac{1}{2}\left[-\frac{\cos 3x}{3} + \cos x\right] + C$

$\_\_\_ = -\frac{\cos 3x}{6} + \frac{\cos x}{2} + C$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**5** Find: **(a)** $\int \sin^2 2x\,dx$ **(b)** $\int \cos^3 2x\,dx$ **(c)** $\int \sin^2 2x\cos^2 2x\,dx$

a) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

so $\int \sin^2 2x\,dx = \int \left(\frac{1-\cos 4x}{2}\right)dx = \frac{1}{2}\left[\int dx - \int \cos 4x\,dx\right]$

—— $= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + C = \frac{x}{2} - \frac{\sin 4x}{8} + C$

b) $\int \cos^3 2x\,dx = \int \cos 2x\left(\cos^2 2x\right)dx = \int \cos 2x\left(1-\sin^2 2x\right)dx$

—— $= \int \cos 2x\,dx - \int \sin^2 2x\,\cos 2x\,dx$

—— $= \int \cos 2x\,dx - \int \sin^2 2x\left(\frac{\sin 2x}{2}\right)'dx$

—— $= \int \cos 2x\,dx - \frac{1}{2}\int \sin^2 2x\,(\sin 2x)'\,dx$

—— $= \frac{\sin 2x}{2} - \frac{1}{2}\,\frac{\sin^3 2x}{3} + C$

—— $= \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C$

—— $= \frac{\sin 2x}{2}\left[1 - \frac{\sin^2 2x}{3}\right] + C$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**5** Find: (g) $\int \cos^5 x\, dx$ ✱ $\int \cos^4 x \sin^3 x\, dx$

g/ $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx$

$= \int [1 - 2\sin^2 x + \sin^4 x] \cos x\, dx$

$= \int \cos x\, dx - 2\int \sin^2 x \cos x\, dx + \int \sin^4 x \cos x\, dx$

$= \sin x - 2\int \sin^2 x\, (\sin x)'\, dx + \int \sin^4 x\, (\sin x)'\, dx$

$= \sin x - 2\,\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$

✱ $\int \cos^4 x \sin^3 x\, dx = \int \cos^4 x \times \sin x \times \sin^2 x\, dx$

$= \int \cos^4 x \times \sin x \times (1-\cos^2 x)\, dx$

$= \int \cos^4 x \times \sin x\, dx - \int \cos^6 x \sin x\, dx$

using the change of variable: $u = \cos x$ so $\frac{du}{dx} = -\sin x$

or $du = -\sin x\, dx$

$= -\int u^4\, du + \int u^6\, du$

$= -\frac{u^5}{5} + \frac{u^7}{7} + C$

$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

# INTEGRATION OF TRIGONOMETRIC FUNCTIONS

**7** Evaluate: **(d)** $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$ **(e)** $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$ **(f)** $\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x \, dx$

d) Note that if $f(x) = \sin^3 x \cos x$

then, $f(-x) = \sin^3(-x)\cos(-x) = -\sin^3 x \cos x = -f(x)$

So $f$ is <u>odd</u>. So between two opposite values ($-\pi$ and $\pi$) around the origin, the integral is zero. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx = 0$

<u>Alternatively</u>:

$$\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx = \int_{-\pi}^{\pi} \sin^3 x \, (\sin x)' \, dx = \left[\frac{\sin^4 x}{4}\right]_{-\pi}^{\pi} = \frac{1}{4}\left[\sin^4 \pi - \sin^4(-\pi)\right] = 0$$

e) $$\int_0^{\pi/4} \tan x \sec^2 x \, dx = \int_0^{\pi/4} \tan x \, (\tan x)' \, dx = \left[\frac{\tan^2 x}{2}\right]_0^{\pi/4}$$

$$= \frac{\tan^2(\pi/4)}{2} - \frac{\tan^2 0}{2} = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

f) $$\int_{\pi}^{3\pi/2} \sin x \cos x \, dx = \int_{\pi}^{3\pi/2} \frac{\sin 2x}{2} \, dx = \frac{1}{2}\int_{\pi}^{3\pi/2} \sin 2x \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_{\pi}^{3\pi/2} = -\frac{1}{2}\left[\frac{\cos 3\pi}{2} - \frac{\cos 2\pi}{2}\right]$$

$$= -\frac{1}{2}\left[\frac{-1}{2} - \frac{1}{2}\right]$$

$$= -\frac{1}{2} \times (-1)$$

$$= \frac{1}{2}$$

**7** Evaluate: **(g)** $\int_0^{\pi} 2\sin\theta\cos^2\theta\,d\theta$ **(h)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(x-\frac{\pi}{4}\right)dx$

g) $\int_0^{\pi} 2\sin\theta\cos^2\theta\,dx = 2\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = 2\int_0^{\pi}\cos^2\theta\,(-\cos\theta)'\,d\theta$

$\_\_ = -2\int_0^{\pi}\cos^2\theta\,(\cos\theta)'\,d\theta = -2\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$

$\_\_ = -\frac{2}{3}\left[\cos^3\theta\right]_0^{\pi} = -\frac{2}{3}\left[\cos^3\pi - \cos^3 0\right]$

$\_\_ = -\frac{2}{3}\left[(-1)^3 - 1^3\right] = -\frac{2}{3}(-1-1) = \frac{4}{3}$

h) $\int_{-\pi/2}^{\pi/2}\cos^2\left(x-\frac{\pi}{4}\right)dx = \int_{-\pi/2}^{\pi/2}\left[\frac{1+\cos 2(x-\pi/4)}{2}\right]dx$

$\_\_ = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[1+\cos\left(2x-\pi/2\right)\right]dx$

$\_\_ = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[1+\cos\left(\frac{\pi}{2}-2x\right)\right]dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[1+\sin 2x\right]dx$

$\_\_ = \frac{1}{2}\left[x - \frac{\cos 2x}{2}\right]_{-\pi/2}^{\pi/2}$

$\_\_ = \frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{1}{2}\times\cos\pi\right)-\left(\frac{-\pi}{2}-\frac{1}{2}\cos(-\pi)\right)\right]$

$\_\_ = \frac{1}{2}\left[2\times\frac{\pi}{2} - \frac{(-1)}{2} + \frac{1}{2}(-1)\right] = \frac{1}{2}\left[\pi + \frac{1}{2} - \frac{1}{2}\right] = \frac{\pi}{2}$