FURTHER APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

1 Find the derivative of $\log_e(\cos x)$.

Using the Chain rule

$$\frac{d}{dx}\left(\ln(\cos x)\right) = \frac{1}{\cos x} \times (-\sin x) = -\tan x$$

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2 Find the equation of the tangent to the curve $y = \tan x$ at $x = \frac{\pi}{4}$.

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \times \cos x - (-\sin x)\sin x}{\cos^2 x} = \sec^2 x$$

So at $x = \frac{\pi}{4}$, the gradient is $\sec^2\frac{\pi}{4} = \frac{1}{\cos^2(\pi/4)} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{4}{2} = 2$

The tangent goes through the point $x = \frac{\pi}{4}$, $\tan\frac{\pi}{4} = 1$

So the equation of the tangent is:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 2x + 1 - \pi/2$$

3 For $f(x) = \sin x + \cos x$ over the domain $0 \leq x \leq 2\pi$, find:

(a) $f'(x)$ (b) $f''(x)$ (c) the coordinates of any turning points
(d) the coordinates of any points of inflection (e) the maximum value of $f(x)$.

a) $f'(x) = \cos x - \sin x$ b) $f''(x) = -\sin x - \cos x$

c) $f'(x) = 0$ when $\cos x - \sin x = 0$ or $\cos x = \sin x$

or $\tan x = 1$ so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$

When $x = \frac{\pi}{4}$ $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$ so $\left(\frac{\pi}{4}, \sqrt{2}\right)$

When $x = \frac{5\pi}{4}$ $f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$ so $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$

These are the 2 turning points.

d) For a point of inflection to occur, we must have $f''(x) = 0$

$f''(x) = -\sin x - \cos x = -f(x)$.

$f''(x) = 0 \Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1$

This occurs when $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$

at $x = \frac{3\pi}{4}$ $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0$ so $\left(\frac{3\pi}{4}, 0\right)$

at $x = \frac{7\pi}{4}$ $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$ so $\left(\frac{7\pi}{4}, 0\right)$

These are the 2 points of inflection.

e) There are 2 turning points as shown at c). But only one maximum which is $\left(\frac{\pi}{4}, \sqrt{2}\right)$. So the maximum value of the function is $\sqrt{2}$

4 For $y = e^{\sin x}$, find the equation of the normal to the curve at the point where $x = 0$.

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x} \times (\sin x)' = e^{\sin x} \times \cos x$$

At $x = 0$ the derivative is $e^{\sin 0} \times \cos 0 = 1 \times 1 = 1$

so the normal to the curve has for gradient $m_{\perp} \times 1 = -1$

so $m_{\perp} = -1$

The normal goes through $x = 0$ $f(0) = e^{\sin 0} = 1$ so $(0, 1)$

The equation of the normal is $y - 1 = -1(x - 0)$

or $y = -x + 1$

5 Find all the points on the graph of $y = 2\sin x + \sin^2 x$, $0 \leq x \leq 4\pi$, at which the tangent is horizontal.

The tangent is horizontal when $f'(x) = 0$

$f'(x) = 2\cos x + 2\sin x \times \cos x = 2\cos x(1 + \sin x)$

so $f'(x) = 0$ when $\cos x = 0$, i.e. $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

or when $\sin x = -1$, i.e. $x = \frac{3\pi}{2}$ or $x = \frac{7\pi}{2}$

At $x = \pi/2$ $f(\pi/2) = 2 + 1 = 3$ so point $(\pi/2, 3)$

At $x = \frac{3\pi}{2}$ $f\left(\frac{3\pi}{2}\right) = -2 + 1 = -1$ so point $\left(\frac{3\pi}{2}, -1\right)$

At $x = \frac{5\pi}{2}$ $f\left(\frac{5\pi}{2}\right) = 2 + 1 = 3$ so point $\left(\frac{5\pi}{2}, 3\right)$

At $x = \frac{7\pi}{2}$ $f\left(\frac{7\pi}{2}\right) = -2 + 1 = -1$ so point $\left(\frac{7\pi}{2}, -1\right)$

6 The tide at a point on the WA coast can be modelled using the equation $y = a\cos nt$. At Cable Beach in WA, over two consecutive days, the average difference between high and low tides is 9.0 metres and the average time between high tide and low tide is 6.1 hours.

(a) What is the amplitude of the tide function at Cable Beach?

(b) How much time passes between successive high tides (i.e. the period) and what is the value of n?

(c) Use this information to obtain the tide function and draw its graph.

(d) If the depth of water at low tide is 0.5 metres, what is the depth of the water 1 hour after low tide?

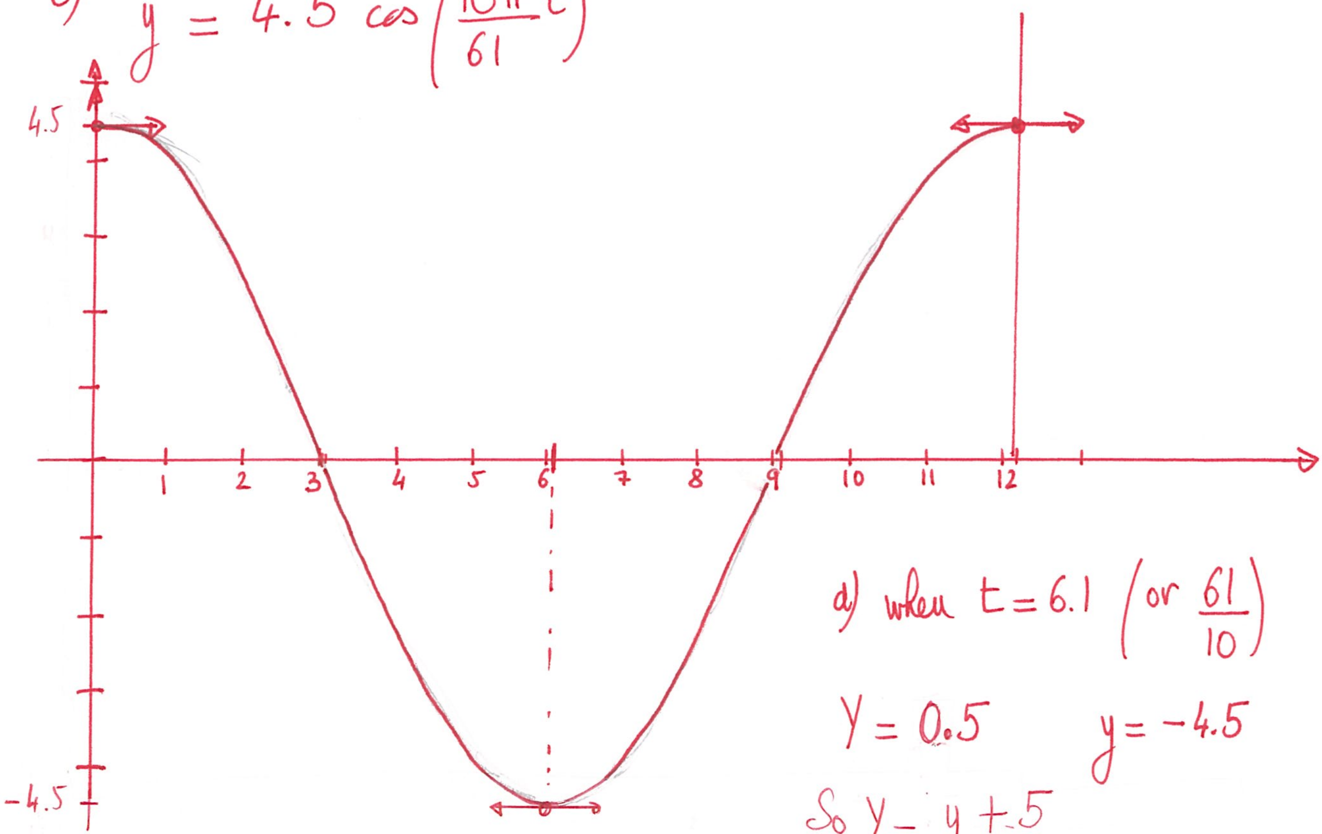
a) The amplitude is half of the average difference between high and low tides, so 4.5 m

b) The average time between high tide and low tide is 6.1 hours. So the time between successive high tides must be 12.2 hours.

$12.2 = \frac{61}{5}$ $\qquad \cos(nt + 2\pi) = \cos[n(t + 12.2)] = \cos(nt + n \times 12.2)$

So we must have $n \times 12.2 = 2\pi$ or $n = \frac{2\pi}{12.2} = \frac{10\pi}{61}$

c) $y = 4.5\cos\left(\frac{10\pi t}{61}\right)$



d) when $t = 6.1$ $\left(\text{or } \frac{61}{10}\right)$

$Y = 0.5$ $\qquad y = -4.5$

So $Y = y + 5$

Y is the real depth

$Y = 5 + 4.5\cos\left(\frac{10\pi t}{61}\right)$

At $t = 6.1 + 1 = 7.1$ $\qquad Y = 5 + 4.5\cos\left(\frac{10\pi \times 7.1}{61}\right) = 1.08$ m

8 Find the equation of the normal to the curve $y = \cot x$ at the point $P\left(\frac{\pi}{4}, 1\right)$.

$f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ $\quad f'(x) = \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$

$f'(x) = -\frac{1}{\sin^2 x} = -\csc^2 x$

At $x = \pi/4$ $\quad f'\left(\frac{\pi}{4}\right) = -\frac{1}{(\sqrt{2}/2)^2} = -\frac{4}{2} = -2$ so $\boxed{m_\perp = \frac{1}{2}}$

At $x = \pi/4$ $\quad f(\pi/4) = \cot(\pi/4) = \frac{\cos \pi/4}{\sin \pi/4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$

So the equation of the normal is

$y - 1 = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$ or $y = \frac{1}{2}x + 1 - \frac{\pi}{8}$

9 If $y = e^{\operatorname{cosec} x}$, find the equation of the tangent to the curve at $x = \frac{\pi}{2}$.

$f(x) = e^{\operatorname{cosec} x}$ $\quad f'(x) = e^{\operatorname{cosec} x} \times (\operatorname{cosec} x)'$

$\operatorname{cosec} x = \frac{1}{\sin x}$ $\quad$ so $(\operatorname{cosec} x)' = \frac{-\cos x}{\sin^2 x} = -\cot x \times \csc x$

$\therefore f'(x) = -e^{\csc x} \times \cot x \times \csc x$

$f'\left(\frac{\pi}{2}\right) = -e^{1/1} \times \frac{(-\cos \pi/2)}{\sin^2 \pi/2} = 0$ $\quad$ horizontal line as $f'\left(\frac{\pi}{2}\right) = 0$

For $x = \frac{\pi}{2}$ $\quad f(x) = e^{\csc \pi/2} = e^{1/\sin \pi/2} = e^{1/1} = e$

So it's the line $y = e$

10 If $y = 3\cos 4x$, prove that $\frac{d^2y}{dx^2} + 16y = 0$.

$$\frac{dy}{dx} = 3 \times (-\sin 4x) \times 4 = -12 \sin 4x$$

$$\frac{d^2y}{dx^2} = -12 \times (\cos 4x) \times 4 = -48 \cos 4x$$

But $\cos 4x = y/3$

so $$\frac{d^2y}{dx^2} = -48 \times \frac{y}{3}$$

or $$\frac{d^2y}{dx^2} = -16y$$

or $$\frac{d^2y}{dx^2} + 16y = 0$$