RECURRENCE RELATIONS

Integration by parts can also be used to find a **recurrence relation** that will allow the evaluation of an integral. Given $I_n = \int_a^b [f(x)]^n \, dx$, you find a relationship of the form $I_n = kI_{n-1}$, where $I_{n-1} = \int_a^b [f(x)]^{n-1} \, dx$. After I_1 or another of the other easy I integrals can be found, the recurrence relation is then used to evaluate the original integral.

Example 29

Find the recurrence relation for $\int \cos^n x \, dx$ and use it to evaluate: $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$

Solution

Let $I_n = \int \cos^n x \, dx$ and write $\cos^n x = \cos^{n-1} x \times \cos x$:

$$I_n = \int \cos^{n-1} x \cos x \, dx$$

Let $u = \cos^{n-1} x$, $\frac{dv}{dx} = \cos x$. This gives $\frac{du}{dx} = (n-1)\cos^{n-2} x \times (-\sin x)$, $v = \sin x$.

Hence: $I_n = \int \cos^n x \, dx = \cos^{n-1} x \times \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx$

$$I_n = \cos^{n-1} x \sin x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x \, dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)I_n$$

$$(1 + n - 1)I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

This relation between I_n and I_{n-2} is a recurrence relation.

To find $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$, write: $I_4 = \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$

From the recurrence relation for $n = 4$, we have: $I_4 = \left[\frac{\cos^3 x \sin x}{4}\right]_0^{\frac{\pi}{2}} + \frac{3}{4} I_2 = \frac{3}{4} I_2$

and $I_2 = \left[\frac{\cos x \sin x}{2}\right]_0^{\frac{\pi}{2}} + \frac{1}{2} I_0 = 0 + \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$

Thus: $I_2 = \frac{1}{2}[x]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ and so $I_4 = \frac{3}{4} I_2 = \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}$

Hence: $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3\pi}{16}$

Example 30

(a) Let $I_n = \int_0^x \tan^n t\,dt$ where $0 \le x < \frac{\pi}{2}$. Show that: $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

(b) Hence find the exact value of: $\int_0^{\frac{\pi}{4}} \tan^5 t\,dt$

Solution

(a) Write $\tan^n t = \tan^2 t \times \tan^{n-2} t = (\sec^2 t - 1)\tan^{n-2} t$. Thus:

$$I_n = \int_0^x \tan^n t\,dt = \int_0^x (\sec^2 t - 1)\tan^{n-2} t\,dt$$

$$= \int_0^x \sec^2 t\,\tan^{n-2} t\,dt - \int_0^x \tan^{n-2} t\,dt$$

$$= \int_0^x \sec^2 t\,\tan^{n-2} t\,dt - I_{n-2}$$

Now consider: $\int_0^x \sec^2 t\,\tan^{n-2} t\,dt$

It is tempting to try integration by parts here, but using the substitution $u = \tan t$ is much easier.
Let $u = \tan t$ so that $du = \sec^2 t\,dt$. Limits are $t = 0$: $u = 0$ $t = x$: $u = \tan x$

$$\therefore \int_0^x \sec^2 t\,\tan^{n-2} t\,dt = \int_0^{\tan x} u^{n-2}\,du$$

$$= \left[\frac{u^{n-1}}{n-1}\right]_0^{\tan x}$$

$$= \frac{1}{n-1}\tan^{n-1} x$$

Thus: $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

(b) $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 t\,dt$ $\therefore I_5 = \frac{1}{4}\left[\tan^4 x\right]_0^{\frac{\pi}{4}} - I_3 = \frac{1}{4} - I_3$

and $I_3 = \frac{1}{2}\left[\tan^2 x\right]_0^{\frac{\pi}{4}} - I_1 = \frac{1}{2} - I_1$

Now: $I_1 = \int_0^{\frac{\pi}{4}} \tan t\,dt = -\left[\log_e(\cos t)\right]_0^{\frac{\pi}{4}} = -\left(\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e 1\right) = \frac{1}{2}\log_e 2$

Hence: $I_3 = \frac{1}{2} - \frac{1}{2}\log_e 2$ and $I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log_e 2\right) = \frac{1}{2}\log_e 2 - \frac{1}{4}$

This gives: $\int_0^{\frac{\pi}{4}} \tan^5 t\,dt = \frac{1}{2}\log_e 2 - \frac{1}{4}$

Example 31

Obtain a recurrence relation for $\int e^{ax}\cos bx\,dx$ and use it to find $\int e^{3x}\cos 4x\,dx$.

(This example is just outside the scope of this course, but it is worth looking at as a demonstration of just how useful this process can be.)

Solution

Write: $I = \int e^{ax}\cos bx\,dx$

Let $u = e^{ax}$, $\frac{dv}{dx} = \cos bx$. This gives $\frac{du}{dx} = ae^{ax}$, $v = \frac{1}{b}\sin bx$.

$\therefore\ I = \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$

Now write: $H = \int e^{ax}\sin bx\,dx$

Let $u = e^{ax}$, $\frac{dv}{dx} = \sin bx$. This gives $\frac{du}{dx} = ae^{ax}$, $v = -\frac{1}{b}\cos bx$.

$\therefore\ H = \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$

Substitute this expression for H into the expression for I:

$$I = \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left(-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right)$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$\therefore\ I = \int e^{ax}\cos bx\,dx = \frac{b^2}{b^2 + a^2}\left(\frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx\right) + C$$

$$= \frac{e^{ax}}{b^2 + a^2}(b\sin bx + a\cos bx) + C$$

In $\int e^{3x}\cos 4x\,dx$ you have $a = 3$ and $b = 4$, thus: $\int e^{3x}\cos 4x\,dx = \frac{e^{3x}}{25}(4\sin 4x + 3\cos 4x) + C$