POLYNOMIALS

A real polynomial $P(x)$ is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $n, n-1, \ldots$ are all positive integers and $a_n, a_{n-1}, \ldots$ are the coefficients, which for convenience will usually be chosen as real integers.

The term $a_n x^n$ is the **leading term**. If $a_n \neq 0$, the polynomial is said to be of the n-th degree, or 'degree n'.

e.g. $P(x) = 4x^5 - 2x^4 + x^2 + 5x - 6$ is a polynomial of degree 5.

A real polynomial $P(x)$ is defined for all real x and is a continuous and differentiable function of x.

$P(x) = 0$ is a polynomial equation of degree n. Real numbers x that satisfy this polynomial equation are called the real **roots** of the equation or the real **zeros** of the corresponding polynomial.

In the Mathematics Advanced course (see *New Senior Mathematics Advanced for Years 11 & 12*) you have already investigated the roots of the quadratic equation $ax^2 + bx + c = 0$ and found that there are at most two real roots, but there may be only one or none.

Similarly, any polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ will have at most n real roots. (Finding these real roots will become easier after you study the factor theorem.)

- $2x^3 + 4x^2 - 3x - 5$ is a polynomial of the 3rd degree in x, because the highest power of x is 3.
- $x^4 - 2x + 7$ is a polynomial of the 4th degree in x.
- $x^{\frac{1}{2}} + 2 + x$ and $\frac{1}{x} + 2x^{-3}$ are not polynomials, because they include powers of x that are not positive integers.

When the leading coefficient $a_n = 1$, the polynomial is said to be **monic**. $x^4 - 2x + 7$ is a monic polynomial.
A polynomial can also be described according to the subset of real numbers that contain the coefficients $a_n, a_{n-1}, \ldots, a_0$.

$2x^2 - 7x + 1$ is a polynomial of degree 2 ("quadratic polynomial") whose coefficients are integers

$\frac{1}{4}x^5 + 2x + \frac{3}{8}$ is a polynomial of degree 5 whose coefficients are rational numbers

$\sqrt{3}x - \sqrt{6}$ is a polynomial of degree 1 whose coefficients are irrational numbers

$4x^2 + \pi x - \frac{1}{2}$ is a polynomial of degree 2 whose coefficients are irrational numbers

Example 1

Express the polynomial $P(x) = x^3 - x^4 + 6x^2$ in standard form. Then write:

(a) its degree **(b)** the constant term **(c)** the coefficient of x^2 **(d)** the leading term

(e) the greatest number of real zeros possible. **(f)** Hence solve the equation $P(x) = 0$.

Solution

$P(x) = -x^4 + x^3 + 6x^2$. In standard form (fully expanded): $P(x) = -x^4 + x^3 + 6x^2 + 0x + 0$

(a) degree = 4 **(b)** constant term = 0 **(c)** coefficient of $x^2 = 6$ **(d)** leading term $= -x^4$

(e) The polynomial is of degree 4, so there are at most four zeros.

(f)
$$-x^4 + x^3 + 6x^2 = 0$$
$$-x^2(x^2 - x - 6) = 0$$
$$x^2(x-3)(x+2) = 0$$
$$\therefore\ x = 0, 3, -2$$

The equation has three real roots. $x = 0$ is a double root.

A graph of $y = x^3 - x^4 + 6x^2$ shows that at $x = 0$, the curve touches the x-axis.

Operations with polynomials

You have added, subtracted and multiplied simple algebraic expressions before. With more complicated polynomials the only difference is that more terms are involved, so you must be more careful that you don't miss any terms.

Example 2

If $A(x) = x^2 + 2x + 3$, $B(x) = 2x - 5$, $C(x) = 3x^4 - 3x^2 + 5x + 6$ and $D(x) = x^5 - 3x^2 + 1$, simplify:

(a) $A(x) + C(x)$ **(b)** $B(x) \times D(x)$ **(c)** $D(x) - C(x)$ **(d)** $A(x) \times D(x)$ **(e)** $A(x) + 2C(x) - 3B(x)$

Solution

(a) $A(x) + C(x) = x^2 + 2x + 3 + 3x^4 - 3x^2 + 5x + 6$
$= 3x^4 - 2x^2 + 7x + 9$

(b) $B(x) \times D(x) = (2x - 5)(x^5 - 3x^2 + 1)$
$= 2x^6 - 6x^3 + 2x - 5x^5 + 15x^2 - 5$
$= 2x^6 - 5x^5 - 6x^3 + 15x^2 + 2x - 5$

(c) $D(x) - C(x) = x^5 - 3x^2 + 1 - (3x^4 - 3x^2 + 5x + 6)$
$= x^5 - 3x^2 + 1 - 3x^4 + 3x^2 - 5x - 6$
$= x^5 - 3x^4 - 5x - 5$

(d) $A(x) \times D(x) = (x^2 + 2x + 3)(x^5 - 3x^2 + 1)$
$= x^7 - 3x^4 + x^2 + 2x^6 - 6x^3 + 2x + 3x^5 - 9x^2 + 3$
$= x^7 + 2x^6 + 3x^5 - 3x^4 - 6x^3 - 8x^2 + 2x + 3$

(e) $A(x) + 2C(x) - 3B(x) = x^2 + 2x + 3 + 2(3x^4 - 3x^2 + 5x + 6) - 3(2x - 5)$
$= x^2 + 2x + 3 + 6x^4 - 6x^2 + 10x + 12 - 6x + 15$
$= 6x^4 - 5x^2 + 6x + 30$