INTEGRATION OF $sin^2 x$ and $cos^2 x$

$\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ cannot be found without using a substitution. However, a useful substitution comes from the formulae for $\cos 2x$. This may not be given in a question, but you must be able to use it when necessary.

The important results are as follows: $\cos 2x = \cos^2 x - \sin^2 x$

Using $\cos^2 x + \sin^2 x = 1$: $\cos 2x = 2\cos^2 x - 1$

$\therefore \quad \cos^2 x = \dfrac{1+\cos 2x}{2}$

Thus:
$$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$$
$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Similarly: $\cos 2x = 1 - 2\sin^2 x$

$\therefore \quad \sin^2 x = \dfrac{1-\cos 2x}{2}$

Thus:
$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 13

(a) Find $3\int \sin^2 x\,dx$. **(b)** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx$. **(c)** Find $\int \cos^2 3x\,dx$.

Solution

(a)
$$3\int \sin^2 x\,dx = 3 \times \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{3}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{3x}{2} - \frac{3\sin 2x}{4} + C$$

(b)
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (1+\cos 2x)\,dx$$
$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right)\right]$$
$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} - 0\right)$$
$$= \frac{1}{2}\left(\frac{3\pi}{4} + \frac{1}{2}\right) = \frac{1}{8}(3\pi + 2)$$

(c)
$$\int \cos^2 3x\,dx = \frac{1}{2}\int (1+\cos 6x)\,dx$$
$$= \frac{1}{2}\left(x + \frac{\sin 6x}{6}\right) + C$$
$$= \frac{x}{2} + \frac{\sin 6x}{12} + C$$

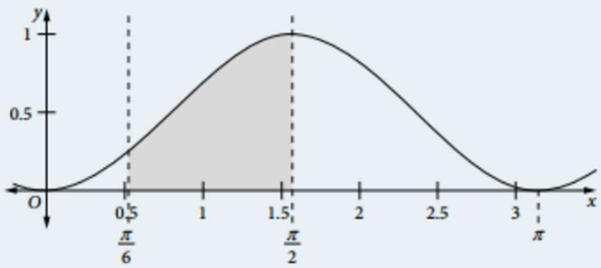
INTEGRATION OF $sin^2 x$ and $cos^2 x$

Example 14

Find the area bounded by the curve $y = \sin^2 x$, the x-axis and the ordinates at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Solution

Sketch the function.



$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin \pi - \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0 - \frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right)$$

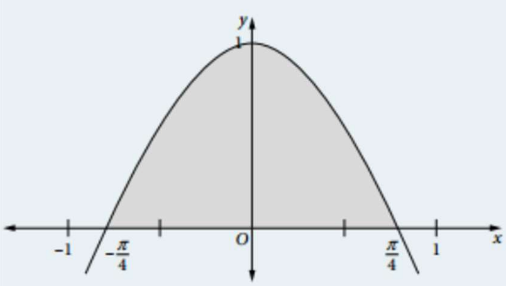
$$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{4\pi + 3\sqrt{3}}{24} \text{ square units}$$

Example 15

The region under the curve $y = \cos 2x$ between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the volume of the solid of revolution formed.

Solution

Sketch the function.



$$\text{Volume} = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2\, dx$$

$$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2x\, dx$$

$$= \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4x)\, dx$$

$$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} + 0 - \left(-\frac{\pi}{4} + 0\right)\right] = \frac{\pi^2}{4} \text{ cubic units}$$