TRIGONOMETRIC IDENTITIES AND PROOFS

Because the equation of the unit circle is: $x^2 + y^2 = 1$

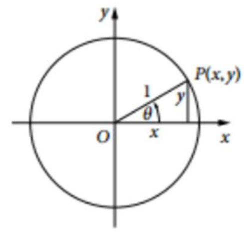
and, by definition: $x = \cos\theta,\ y = \sin\theta$

it follows that:

$$\cos^2\theta + \sin^2\theta = 1 \quad [1]$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta$$

Dividing [1] by $\cos^2\theta$: $1 + \tan^2\theta = \sec^2\theta$ [2]

Dividing [1] by $\sin^2\theta$: $\cot^2\theta + 1 = \mathrm{cosec}^2\theta$ [3]



Statements [1], [2] and [3] are called **identities** as they are true for all values of θ.

Example 11

If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact values of $\cos\theta$ and $\tan\theta$.

Solution

Using: $\cos^2\theta + \sin^2\theta = 1$

$\sin\theta = \frac{3}{5}$: $\cos^2\theta + \frac{9}{25} = 1$

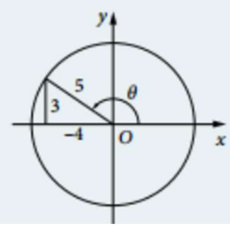
$$\cos^2\theta = \frac{16}{25}$$

So: $\cos\theta = \pm\frac{4}{5}$; because it is given that $\frac{\pi}{2} < \theta < \pi$, we have $\cos\theta = -\frac{4}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{-\frac{4}{5}}$$

$$\tan\theta = -\frac{3}{4}$$

Alternatively: in a right-angled triangle with a side length of 3 and a hypotenuse length of 5, you can calculate that the third side length is 4. You also know from $\frac{\pi}{2} < \theta < \pi$ that the angle is in the second quadrant, so the x value will be negative. Hence from $\sin\theta = \frac{3}{5}$ you can obtain $\cos\theta = -\frac{4}{5}$ and $\tan\theta = -\frac{3}{4}$.



Example 12

Simplify:

(a) $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$ **(b)** $1 + \tan^2(\frac{\pi}{2} - \theta)$ **(c)** $\sqrt{a^2 + x^2}$, for $x = a\tan\theta$ and $0 < \theta < \frac{\pi}{2}$

Solution

(a)
$$\frac{1-\cos^2\theta}{1-\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

(b)
$$1 + \tan^2(\tfrac{\pi}{2} - \theta) = 1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

(c)
$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$$

Example 13

Prove the following identities.

(a) $\dfrac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta}=\cos^2\theta$ **(b)** $\dfrac{\cos\theta}{1-\sin\theta}-\tan\theta=\sec\theta$ **(c)** $\tan\theta\sin\theta+\cos\theta=\sec\theta$

Solution

(a) $\text{LHS}=\dfrac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta}$ Use $\cos^2\theta=1-\sin^2\theta$ and $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{\cos^2\theta}{1}$

$=\cos^2\theta$

$=\text{RHS}$

(b) $\text{LHS}=\dfrac{\cos\theta}{1-\sin\theta}-\tan\theta$ Replace $\tan\theta$ with $\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{\sin\theta}{\cos\theta}$ Find common denominator

$=\dfrac{\cos^2\theta-\sin\theta(1-\sin\theta)}{(1-\sin\theta)\cos\theta}$ Expand

$=\dfrac{\cos^2\theta-\sin\theta+\sin^2\theta}{(1-\sin\theta)\cos\theta}$ Use $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{1-\sin\theta}{(1-\sin\theta)\cos\theta}$ Cancel common factor

$=\dfrac{1}{\cos\theta}$ for $\sin\theta\neq 1$

$=\sec\theta$

$=\text{RHS}$

(c) $\text{LHS}=\tan\theta\sin\theta+\cos\theta$

$=\dfrac{\sin\theta}{\cos\theta}\times\sin\theta+\cos\theta$ Replace $\tan\theta$ with $\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\sin^2\theta}{\cos\theta}+\dfrac{\cos\theta}{1}$ Find common denominator

$=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta}$

$=\dfrac{1}{\cos\theta}$ Use $\sin^2\theta+\cos^2\theta=1$

$=\sec\theta$

$=\text{RHS}$