USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

1 Find: **(a)** $\int \frac{2\,dx}{(x-3)(x-1)}$

$$\frac{2}{(x-3)(x-1)} = \frac{a}{x-3} + \frac{b}{x-1} = \frac{x(a+b)-a-3b}{(x-3)(x-1)}$$

$$\therefore \begin{cases} a+b=0 \\ -a-3b=2 \end{cases} \Leftrightarrow \begin{cases} b=-a \\ -a-3\times(-a)=2 \end{cases}$$ so $2a=2$ $\boxed{a=1}$ and $\boxed{b=-1}$

$$\therefore \int \frac{2\,dx}{(x-3)(x-1)} = \int \left[\frac{1}{x-3} - \frac{1}{x-1}\right] dx$$

$$\text{——} = \int \frac{dx}{x-3} - \int \frac{dx}{x-1}$$

$$\text{——} = \ln|x-3| - \ln|x-1| + C$$

$$\text{——} = \ln\left|\frac{x-3}{x-1}\right| + C$$

USING PARTIAL FRACTIONS TO FIND INTEGRALS

1 Find: **(f)** $\int \frac{(2x-5)dx}{(x-3)(x-2)}$

$$\frac{2x-5}{(x-3)(x-2)} = \frac{a}{x-3} + \frac{b}{x-2} = \frac{x(a+b)+(-3b-2a)}{(x-3)(x-2)}$$

$$\therefore \begin{cases} a+b=2 \\ -3b-2a=-5 \end{cases} \Leftrightarrow \begin{cases} b=2-a \\ -3(2-a)-2a=-5 \end{cases}$$

so $-6+3a-2a=-5$ or $a=-5+6=1$

so $a=1$ and $b=2-1=1$

$$\therefore \int \frac{2x-5}{(x-3)(x-2)}dx = \int \left[\frac{1}{x-3} + \frac{1}{x-2}\right] dx$$

$$\text{———} = \int \frac{dx}{x-3} + \int \frac{dx}{x-2}$$

$$\text{———} = \ln|x-3| + \ln|x-2| + C$$

$$\text{———} = \ln|(x-3)(x-2)| + C$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

1 Find: **(e)** $\int \frac{x}{(x^2+4)(x^2+5)}dx$

$$\frac{x}{(x^2+4)(x^2+5)} = \frac{ax+b}{x^2+4} + \frac{cx+d}{x^2+5}$$

$$\text{———} = \frac{x^3(a+c)+x^2(b+d)+x(5a+4c)+(4d+5b)}{(x^2+4)(x^2+5)}$$

So $\begin{cases} a+c=0 \\ b+d=0 \\ 5a+4c=1 \\ 4d+5b=0 \end{cases} \Leftrightarrow \begin{cases} c=-a \\ d=-b \\ 5a+4(-a)=1 \\ 4(-b)+5b=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ c=-1 \\ b=0 \\ d=0 \end{cases}$

$$\therefore \int \frac{x\,dx}{(x^2+4)(x^2+5)} = \int\left[\frac{x}{x^2+4} - \frac{x}{x^2+5}\right]dx$$

$$\text{———} = \int \frac{x}{x^2+4}dx - \int \frac{x}{x^2+5}dx$$

$$\text{———} = \frac{1}{2}\int \frac{2x}{x^2+4}dx - \frac{1}{2}\int \frac{2x}{x^2+5}dx$$

$$\text{———} = \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\ln|x^2+5| + C$$

$$\text{———} = \ln\sqrt{\frac{x^2+4}{x^2+5}} + C$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

7 Evaluate: (a) $\int_4^5 \frac{x^2-5}{x^2-2x-3}\,dx$

$\Delta = 4 - 4\times(-3) = 16$

$x_1 = \frac{2-4}{2} = -1 \qquad x_2 = \frac{2+4}{2} = 3$

so $x^2 - 2x - 3 = (x+1)(x-3)$.

$$\frac{x^2-5}{x^2-2x-3} = \frac{(x^2-2x-3)+2x+3-5}{x^2-2x-3} = 1 + \frac{2x-2}{x^2-2x-3}$$

$$\text{——} = 1 + \frac{2(x-1)}{(x+1)(x-3)}$$

we work on the 2nd term.

$$\frac{2(x-1)}{(x+1)(x-3)} = \frac{a}{x+1} + \frac{b}{x-3} = \frac{x(a+b)+(b-3a)}{(x-1)(x-3)}$$

so $\begin{cases} a+b = 2 \\ b-3a = -2 \end{cases} \Leftrightarrow \begin{cases} b = 2-a \\ (2-a)-3a = -2 \end{cases} \Leftrightarrow \begin{cases} -4a = -4 \\ b = 2-1 = 1 \end{cases} \quad \boxed{a=1} \quad \boxed{b=1}$

$$\therefore \int \frac{x^2-5}{x^2-2x-3}\,dx = \int \left[1 + \frac{1}{x+1} + \frac{1}{x-3}\right] dx$$

$$\text{——} = x + \ln|x+1| + \ln|x-3| + C.$$

$$\text{——} = x + \ln|(x+1)(x-3)| + C.$$

$$\therefore \int_4^5 \frac{x^2-5}{x^2-2x-3}\,dx = \Big[x + \ln|(x+1)(x-3)|\Big]_4^5$$

$$\text{——} = 1 + \ln|6\times 2| - \ln|5\times 1| = 1 + \ln\left|\frac{12}{5}\right|$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

8 Given that $\dfrac{1}{(x+2)(x+1)^2} = \dfrac{1}{x+2} - \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2}$, evaluate: $\displaystyle\int_0^1 \frac{1}{(x+2)(x+1)^2}\,dx$

$$\int_0^1 \frac{dx}{(x+2)(x+1)^2} = \int_0^1 \left[\frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\right] dx$$

$$\text{——} = \int_0^1 \frac{dx}{x+2} - \int_0^1 \frac{dx}{x+1} + \int_0^1 \frac{dx}{(x+1)^2}$$

$$\text{——} = \Big[\ln|x+2|\Big]_0^1 - \Big[\ln|x+1|\Big]_0^1 + \int_0^1 \frac{dx}{(x+1)^2}$$

$$\text{——} = \ln\left|\frac{3}{2}\right| - \ln 2 + \int_0^1 \frac{dx}{(x+1)^2}$$

$$\text{——} = \ln\left(\frac{3}{4}\right) + \int_0^1 \frac{dx}{(x+1)^2}$$

Let $u = x+1$ so $\dfrac{du}{dx} = 1$ or $du = dx$

$$\int_0^1 \frac{dx}{(x+1)^2} = \int_1^2 \frac{du}{u^2} = \int_1^2 u^{-2}\,du = \left[\frac{u^{-2+1}}{-2+1}\right]_1^2 = \left[\frac{u^{-1}}{-1}\right]_1^2$$

$$\text{——} = \left[-\frac{1}{u}\right]_1^2 = \left[\frac{1}{u}\right]_2^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \int_0^1 \frac{dx}{(x+2)(x+1)^2} = \ln\left(\frac{3}{4}\right) + \frac{1}{2}$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

21 Using an appropriate substitution of the type $t = \tan x$, find: $\displaystyle\int \frac{dx}{\sin 2x - \cos 2x}$

if $t = \tan\frac{x}{2}$, then $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

So with $t = \tan x$ $\quad \sin 2x = \dfrac{2t}{1+t^2}$ and $\cos 2x = \dfrac{1-t^2}{1+t^2}$

$$I = \int \frac{dx}{\sin 2x - \cos 2x} = \int \frac{1}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2}$$

as $\dfrac{dt}{dx} = \sec^2 x = \dfrac{1}{\cos^2 x} = \dfrac{1}{\frac{\cos 2x + 1}{2}} = \dfrac{2}{\frac{1-t^2}{1+t^2} + 1}$

so $\dfrac{dt}{dx} = \dfrac{2(1+t^2)}{1-t^2+1+t^2} = 1+t^2$ $\quad$ so $dx = \dfrac{dt}{1+t^2}$

$$I = \int \frac{dt}{2t - (1-t^2)} = \int \frac{dt}{t^2+2t-1} \qquad \Delta = 4 - 4\times(-1) = 8 = (2\sqrt{2})^2$$

$t_1 = \dfrac{-2-2\sqrt{2}}{2} = -1-\sqrt{2}$ and $t_2 = -1+\sqrt{2}$

So $t^2+2t-1 = (t+1+\sqrt{2})(t+1-\sqrt{2})$

$$\frac{1}{t^2+2t-1} = \frac{a}{t+1+\sqrt{2}} + \frac{b}{t+1-\sqrt{2}} = \frac{t(a+b) + (a-\sqrt{2}a+b+\sqrt{2}b)}{t^2+2t-1}$$

$\therefore \begin{cases} a+b=0 \\ a-\sqrt{2}a+b+\sqrt{2}b=1 \end{cases} \Leftrightarrow \begin{cases} b=-a \\ \cancel{a}-\sqrt{2}a\cancel{-a}-\sqrt{2}a=1 \end{cases} \Leftrightarrow \begin{cases} b=-a \\ -2\sqrt{2}a=1 \end{cases} \begin{cases} a=-\frac{1}{2\sqrt{2}} \\ b=\frac{1}{2\sqrt{2}} \end{cases}$

$$I = \int \frac{-\frac{1}{2\sqrt{2}}}{t+1+\sqrt{2}}\,dt + \int \frac{\frac{1}{2\sqrt{2}}\,dt}{t+1-\sqrt{2}} = -\frac{1}{2\sqrt{2}}\ln|t+1+\sqrt{2}| + \frac{1}{2\sqrt{2}}\ln|t+1-\sqrt{2}| + C$$

$$I = \frac{1}{2\sqrt{2}}\ln\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x+1-\sqrt{2}}{\tan x+1+\sqrt{2}}\right| + C$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

22 Use the substitution $t = \tan\frac{x}{2}$ to find $\int \frac{dx}{1+3\sin x}$. $= I$

with $t = \tan\frac{x}{2}$ $\quad \sin x = \frac{2t}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$

and $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{\cos x + 1} = \frac{1}{\frac{1-t^2}{1+t^2}+1}$

$\frac{dt}{dx} = \frac{1+t^2}{1-t^2+1+t^2} = \frac{1+t^2}{2}$ so $dx = \frac{2}{1+t^2}\,dt$.

$$I = \int \frac{1}{1+\frac{3\times 2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt = \int \frac{2\,dt}{1+t^2+6t} = \int \frac{2dt}{t^2+6t+1}$$

$\Delta = 36 - 4 = 32 = 2^5 = (4\sqrt{2})^2$ so $t_1 = \frac{-6+4\sqrt{2}}{2} = -3+2\sqrt{2}$

and $t_2 = -3 - 2\sqrt{2}$

So $\frac{2}{t^2+6t+1} = \frac{a}{(t+3-2\sqrt{2})} + \frac{b}{(t+3+2\sqrt{2})}$

$ = \frac{t(a+b)+(3a+2a\sqrt{2}+3b-2b\sqrt{2})}{t^2+6t+1}$

$\therefore \begin{cases} a+b=0 \\ 3a+2a\sqrt{2}+3b-2b\sqrt{2} = 2 \end{cases} \Leftrightarrow \begin{cases} b=-a \\ \cancel{3a}+2a\sqrt{2}-\cancel{3a}+2a\sqrt{2} = 2 \end{cases}$

so $4a\sqrt{2} = 2$ so $a = 1/2\sqrt{2}$ and $b = -\frac{1}{2\sqrt{2}}$

$$\therefore I = \int \frac{1/2\sqrt{2}}{t+3-2\sqrt{2}}\,dt - \int \frac{1/2\sqrt{2}}{t+3+2\sqrt{2}}\,dt$$

$$I = \frac{1}{2\sqrt{2}} \ln\left|\frac{t+3-2\sqrt{2}}{t+3+2\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}} \ln\left|\frac{\tan\left(\frac{x}{2}\right)+3-2\sqrt{2}}{\tan\left(\frac{x}{2}\right)+3+2\sqrt{2}}\right| + C$$

USING **PARTIAL FRACTIONS** TO FIND INTEGRALS

23 Use the substitution $t = \tan x$ to find $\int \frac{1+\sin^2 x}{1+\cos^2 x}\,dx$. $= I$

with $t = \tan x$ $\quad \sin 2x = \frac{2t}{1+t^2}$ $\quad$ and $\cos 2x = \frac{1-t^2}{1+t^2}$

$dx = \frac{dt}{1+t^2}$ $\quad \cos 2\theta = 2\cos^2\theta - 1$ so $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$\cos 2\theta = 1 - 2\sin^2\theta$ so $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\therefore \cos^2 x + 1 = \frac{\cos 2x + 3}{2} = \frac{\frac{1-t^2}{1+t^2} + 3}{2} = \frac{1-t^2+3(1+t^2)}{2(1+t^2)}$

$\therefore \cos^2 x + 1 = \frac{2t^2+4}{2(1+t^2)} = \frac{t^2+2}{1+t^2}$

and $1+\sin^2 x = \frac{3-\cos 2x}{2} = \frac{3 - \frac{1-t^2}{1+t^2}}{2} = \frac{3(1+t^2)-(1-t^2)}{2(1+t^2)}$

$\therefore 1+\sin^2 x = \frac{4t^2+2}{2(1+t^2)} = \frac{2t^2+1}{1+t^2}$

$I = \int \frac{\frac{2t^2+1}{1+t^2}}{\frac{t^2+2}{1+t^2}} \times \frac{dt}{1+t^2} = \int \frac{(2t^2+1)\,dt}{(2+t^2)(1+t^2)}$

$\frac{2t^2+1}{(2+t^2)(1+t^2)} = \frac{at+b}{t^2+2} + \frac{ct+d}{t^2+1} = \frac{t^3(a+c)+t^2(b+d)+t(a+2c)+(b+2d)}{(t^2+2)(t^2+1)}$

So $\begin{cases} a+c=0 \\ b+d=2 \\ a+2c=0 \\ b+2d=1 \end{cases} \Leftrightarrow \begin{cases} c=0 \\ a=0 \\ d=-1 \\ b=2-(-1)=3 \end{cases}$ $\quad I = \int \left[\frac{3}{t^2+2} - \frac{1}{t^2+1}\right] dt$

$I = \frac{3}{\sqrt{2}} \int \frac{\sqrt{2}}{2+t^2}\,dt - \int \frac{dt}{1+t^2} = \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \tan^{-1} t + C$

$I = \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - \tan^{-1}(\tan x) + C = \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - x + C$