THE UNIFORM DISTRIBUTION

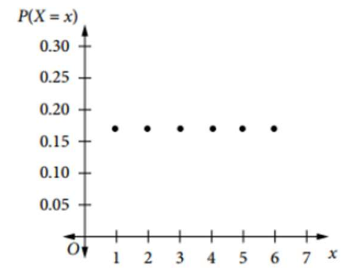
A discrete probability distribution is said to be **uniform** if all values of the random variable are equally likely.

$$P(X = x_i) = \begin{cases} \frac{1}{n} \text{ when } 1 \leq i \leq n \\ 0 \text{ elsewhere} \end{cases}$$

A common example of a uniform distribution is the random variable, X, that is the value of the face showing when a normal, six-sided die is rolled.

$$P(X = x) = \begin{cases} \frac{1}{6} & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{for all other values of } x \end{cases}$$

The graph of this distribution is shown on the right:



The expected value is: $E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$$= \frac{21}{6}$$

$$= 3\frac{1}{2}$$

This can also be determined from the symmetry of the graph of the distribution.

The variance of the distribution is given by $\text{Var}(X) = E(X^2) - [E(X)]^2$:

$$\text{Var}(X) = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) - \left(\frac{7}{2}\right)^2$$

$$= \frac{35}{12}$$

<u>Expected value of a uniform discrete probability distribution</u>

$$E(X) = \sum_{i=1}^{n} x_i \, P(X = x_i) = \sum_{i=1}^{n} x_i \, \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

If the values of x_i range from 1 to n, the expected value is:

$$E(X) = \sum_{i=1}^{n} i \, P(X = x_i) = \sum_{i=1}^{n} i \, \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n}[1 + 2 + 3 + \cdots + (n-1) + n]$$

But: $1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$ see demonstration below[1]

Therefore: $E(X) = \frac{1}{n} \times \frac{n(n+1)}{2}$

$$E(X) = \frac{n+1}{2}$$

[1] Let S be the sum of the first n positive integers. $S = 1 + 2 + 3 + \cdots + (n-1) + n$

We can also write S in descending order: $S = n + (n-1) + \cdots + 3 + 2 + 1$

We add both expressions of S; we can see that each term pairs with the next and each of these sums is $(n+1)$.

Therefore: $2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ times}} = n(n+1)$ i.e. $2S = n(n+1)$ Hence $\boldsymbol{S = \frac{n(n+1)}{2}}$

THE UNIFORM DISTRIBUTION

Variance of a uniform discrete probability distribution:

$Var(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{i=1}^{n} {x_i}^2 \, P(X = x_i) = \sum_{i=1}^{n} {x_i}^2 \, \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2$$

If the values of x_i range from 1 to n, $E(X^2)$ is equal to: $E(X^2) = \frac{1}{n}\sum_{i=1}^{n} i^2$

It can be demonstrated[2] that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Therefore in that case, $E(X^2) = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$

$$Var(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)}{2}\left[\frac{(2n+1)}{3} - \frac{(n+1)}{2}\right]$$

$$Var(X) = \frac{(n+1)}{2}\left[\frac{2(2n+1) - 3(n+1)}{6}\right] = \frac{(n+1)}{2}\left(\frac{n-1}{6}\right)$$

$$Var(X) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

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[2] Let S be the sum of the squares of the first n positive integers. $A_n = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2$

for $n = 1$	$A_1 = 1^2 = 1$
for $n = 2$	$A_2 = 1^2 + 2^2 = 1 + 4 = 5$
for $n = 3$	$A_3 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$
for $n = 4$	$A_4 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

It seems that: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ This indeed is true for $n = 1, n = 2, n = 3$ and $n = 4$, as:

for $n = 1$	$\frac{1(1+1)(2\times 1+1)}{6} = \frac{1\times 2\times 3}{6} = \frac{6}{6} = 1$
for $n = 2$	$\frac{2(2+1)(2\times 2+1)}{6} = \frac{2\times 3\times 5}{6} = \frac{30}{6} = 5$
for $n = 3$	$\frac{3(3+1)(2\times 3+1)}{6} = \frac{3\times 4\times 7}{6} = \frac{84}{6} = 14$
for $n = 4$	$\frac{4(4+1)(2\times 4+1)}{6} = \frac{4\times 5\times 9}{6} = \frac{180}{6} = 30$

We demonstrate this formula by the method of mathematical induction: we assume it is true for n, and we demonstrate that the formula holds for $(n+1)$.

$$A_n = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$A_{n+1} = A_n + (n+1)^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$A_{n+1} = \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} = \frac{(n+1)[2n^2 + n + 6n + 6]}{6} = \frac{(n+1)[2n^2 + 7n + 6]}{6}$$

Factorising the 2nd term, we obtain: $2n^2 + 7n + 6 = (n+2)(2n+3) = (n+2)[2(n+1)+1]$ therefore:

$$A_{n+1} = \frac{(n+1)(n+2)[2(n+1)+1]}{6}$$

So if it is true for A_n, then it is also true for A_{n+1}. As it is true for A_1, we conclude that it must be true for A_2, so it must be true for A_3, so it must be true for A_4, so it must be true for A_5, and so on. So it must be true for any n.

Example 13

A roulette wheel in the United States usually has 38 equal-sized spaces showing the numbers 1 to 36 as well as 0 and 00. When the wheel is spun, a ball will land in one of the 38 spaces at random. For this question assume that 0 and 00 represent the 37th and 38th possible outcomes.

(a) Find the mean of the number of the space the ball lands in.

(b) Find the variance of the number of the space the ball lands in.

Solution

(a) Use the rule for $E(X)$:

$$E(X) = \frac{n+1}{2} = \frac{38+1}{2} = \frac{39}{2} = 19\frac{1}{2}$$

(b) Use the rule for Var (X):

$$\text{Var}(X) = \frac{n^2-1}{12} = \frac{38^2-1}{12} = \frac{1443}{12} = 120\frac{1}{4}$$

This section has focused on uniform distributions where the values of x are 1 to n. However, it is still quite easy to find the expected value and variance for other uniform distributions where the values the distribution takes are consecutive numbers.

This is because such a distribution is a lateral shift of the uniform distribution where x takes the values 1 to n, so the rules $E(X+b) = E(X) + b$ and $Var(X+b) = Var(X)$ can be applied.