ZEROS OF A POLYNOMIAL

1 Given that $(z+2-i)$ is a factor, factorise $z^4+4z^3+3z^2-8z-10$ over:

(a) the set of real numbers (b) the set of complex numbers.

a) The coefficients of $P(z)$ are real, $\therefore$ if $(i-2)$ is a root, then $(-2-i)$ must also be a root

$(z+2-i)(z+2+i) = z^2+4z+4+1 = z^2+4z+5$

$\therefore \quad z^4+4z^3+3z^2-8z-10 = (z^2+4z+5)(z^2-2)$

$= (z^2+4z+5)(z-\sqrt{2})(z+\sqrt{2})$

b) $P(z) = (z+2-i)(z+2+i)(z-\sqrt{2})(z+\sqrt{2})$.

2 Solve the following for z as a complex number.

(a) $z^2-4z+8=0$ (b) $z^3+2z^2-2z+3=0$ (c) $z^6+7z^3-8=0$

a) $\Delta = (-4)^2-4\times 8 = -16 = (4i)^2 \qquad \therefore \quad z = \dfrac{4\pm 4i}{2} = 2\pm 2i$

b) (-3) is an obvious root, $\therefore$ $z^3+2z^2-2z+3 = (z+3)(z^2-z+1)$

For the quadratic, $\Delta = 1-4 = -3 = (\sqrt{3}i)^2$

So the two other roots are $z = \dfrac{1\pm\sqrt{3}i}{2}$

c) We make a change of variable: $X = z^3$. The equation becomes: $X^2+7X-8=0$ which has 1 as an obvious root.

So $X^2+7X-8 = (X-1)(X+8) = [z^3-1][z^3+8]$

$\therefore \quad z^6+7z^3-8 = [z-1][z^2+z+1][z^3+2^3]$

$= [z-1](z^2+z+1)(z+2)(z^2-2z+4)$

For the 1st quadratic, $\Delta = 1-4 = -3 = (\sqrt{3}i)^2$

so 2 solutions $z = \dfrac{-1\pm\sqrt{3}i}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

For the 2nd quadratic $\Delta = 4-4\times 4 = -12 = (2\sqrt{3}i)^2$

so 2 solutions $z = \dfrac{2\pm 2\sqrt{3}i}{2} = 1\pm\sqrt{3}i$

So 6 solutions: $z = 1,\ -2,\ -\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2},\ 1\pm\sqrt{3}i$

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3 Solve $z^3 + z^2 + 3z - 5 = 0$ for z as **(a)** a real number **(b)** a complex number.

$z = 1$ is an obvious solution, $\therefore$ (E) $\Leftrightarrow (z-1)(z^2+2z+5) = 0$

$\Delta = 4 - 4 \times 5 = -16 = (4i)^2$

a) only one real solution, which is $z = 1$

b) $z = 1$ is a solution. The two other solutions are $\dfrac{-2 \pm 4i}{2}$

i.e. $(-1 \pm 2i)$.

4 Solve $z^5 + 3z^4 - z - 3 = 0$ for z as a real number.

$z^5 + 3z^4 - z - 3 = 0 \Leftrightarrow z^5 - z + 3z^4 - 3 = 0$

$\Leftrightarrow z(z^4 - 1) + 3(z^4 - 1) = 0 \Leftrightarrow (z^4-1)(z+3) = 0$

$\Leftrightarrow (z^2-1)(z^2+1)(z+3) = 0 \Leftrightarrow (z-1)(z+1)(z^2+1)(z+3) = 0$

So 3 real solutions: $z = 1$, $z = -1$ and $z = -3$

5 What are the roots of $z^4 - 2z^3 - z + 2 = 0$ for z as a complex number?

A 1, 2 $\boxed{\textbf{B}}$ $1, 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ **C** $-1, -2$ **D** $-1, -2, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$z^4 - 2z^3 - z + 2 = z^4 - z - 2z^3 + 2$

$= z(z^3 - 1) - 2(z^3 - 1)$

$= (z-2)(z^3-1)$

$= (z-2)(z-1)(z^2+z+1)$

$\Delta = 1^2 - 4 = -3 = (\sqrt{3}i)^2$ so 2 solutions for the quadratic,

which are $z = \dfrac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Response $\boxed{B}$

6 Find the values of the real numbers a and b such that $1 + i$ is a root of the equation $z^3 + az + b = 0$.

For $(1+i)$ to be a root of the equation $z^3 + az + b = 0$,

we must have $(1+i)^3 + a(1+i) + b = 0$

$\Leftrightarrow 1 + 3i + 3i^2 + i^3 + a + ai + b = 0$

$\Leftrightarrow 1 + 3i - 3 - i + a + ai + b = 0$

$\Leftrightarrow [-2 + a + b] + i[2 + a] = 0$

Therefore, we must have $\begin{cases} 2 + a = 0 \\ -2 + a + b = 0 \end{cases}$ (both real parts and imaginary part are zero)

$\therefore \quad a = -2 \quad$ and $\therefore \quad b = 4$

8 Solve $3z^3 - 4z^2 - 13z - 6 = 0$ for z if z is a real number.

We try to find an obvious root.

$P(1) = 3 \times 1^3 - 4 \times 1^2 - 13 \times 1 - 6 \neq 0$ so 1 not a root

$P(-1) = 3 \times (-1)^3 - 4 \times (-1)^2 - 13 \times (-1) - 6 = 0$ YES!

So we can factorise by $(z+1)$.

$3z^3 - 4z^2 - 13z - 6 = (z+1)(3z^2 - 7z - 6)$

We now solve the quadratic: $\Delta = 49 + 4 \times 6 \times 3 = 121$

So $z = \dfrac{7 \pm 11}{6}$ i.e., $z = 3$ or $z = -2/3$

3 solutions: -1, 3 and $-\dfrac{2}{3}$

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9 Solve $z^4 - z^3 + 6z^2 - z + 15 = 0$ for z given that $z = 1 - 2i$ is a root of the equation.

The polynomial has real coefficients, $\therefore$ $1+2i$ must also be a root. (by the conjugate roots theorem).

$[z-(1-2i)][z-(1+2i)] = z^2 - 2z + 5$, therefore it factorises as:

$z^4 - z^3 + 6z^2 - z + 15 = (z^2 - 2z + 5)(z^2 + z + 3)$

We now solve the 2nd quadratic: $\Delta = 1 - 4\times 3 = -11 = (\sqrt{11}\,i)^2$

So the two other roots are $z = \dfrac{-1 \pm \sqrt{11}\,i}{2}$

12 Write an equation of the lowest possible degree with **(i)** complex coefficients **(ii)** rational coefficients that includes the following among its roots.

(a) $2, 1+i$ **(b)** $\sqrt{3}+1, 2-i$

a) i) the equation has 2 roots and complex coefficients, $\therefore$ must be of degree 2.

e.g. $[z-2][z-(1+i)] = 0 \iff z^2 + z(-1-i-2) + 2(1+i) = 0$

ii) if the equation has real coefficients, then $(1-i)$ must also be a root. So there are 3 roots, i.e 2, $1+i$ and $1-i$. $\therefore$ the lowest possible degree is 3, and the equation could be:

$(z-2)[z-(1+i)][z-(1-i)] = (z-2)[z^2 - 2z + 2]$

b) i) the equation has 2 roots and complex coefficients $\therefore$ must be of degree 2.

e.g. $[z-(\sqrt{3}+1)][z-(2-i)] = 0$

ii) if the equation has real coefficients, then $(2+i)$ must be also a root (by the conjugate root theorem). Further, for the coefficients to be rational, $(\sqrt{3}-1)$ must also be a root. So 4 roots, i.e. degree 4, it could be: $[z-(\sqrt{3}-1)][z-(\sqrt{3}+1)][z-(2+i)][z-(2-i)] = 0$

$\iff [z^2 - 2z - 2][z^2 - 4z + 5] = 0$

$\iff z^4 + z^3(-4-2) + z^2(5+8-2) + z(-10+8) - 10 = 0$

$\iff z^4 - 6z^3 + 11z^2 - 2z - 10 = 0$

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16 Find the real numbers k such that $z = ki$ is a root of the equation $z^3 + (2+i)z^2 + (2+2i)z + 4 = 0$. Hence, or otherwise, find the three roots of the equation.

(ki) is a root, $\therefore$ $(ki)^3 + (2+i)(ki)^2 + (2+2i)\,ki + 4 = 0$

$\Leftrightarrow k^3 \times (-i) - k^2(2+i) + 2ki - 2k + 4 = 0$

$\Leftrightarrow [-2k^2 - 2k + 4] + i[-k^3 - k^2 + 2k] = 0$

$\Leftrightarrow -2[k^2 + k - 2] - ik[k^2 + k - 2] = 0$

$\therefore$ we must have $k^2 + k - 2 = 0$ $\quad \Delta = 1 + 4 \times 2 = 9 = 3^2$

So either $k = -2$ or $k = 1$

$\therefore$ $(-2i)$ and i are roots of the equation.

$P(z) = (z+2i)(z-i)(z+2)$ $\quad$ the 3 roots are $-2, i, -2i$

17 Solve the following equations using a calculus method.

(a) $z^4 + 4z^3 + 5z^2 + 4z + 4 = 0$, given that it has a root of multiplicity 2. ← i.e must be a root of both $P'(z)$ and $P(z)$.

$P(z) = z^4 + 4z^3 + 5z^2 + 4z + 4$

$P'(z) = 4z^3 + 12z^2 + 10z + 4 = 2[2z^3 + 6z^2 + 5z + 2]$

We look for obvious solutions to $P'(z) = 0$

$P'(1) \neq 0$, $\quad P'(-1) \neq 0$, $\quad P'(-2) = 0$ YES!

We try to see if (-2) is also a solution to $P(z) = 0$

$P(-2) = (-2)^4 + 4(-2)^3 + 5 \times (-2)^2 + 4 \times (-2) + 4 = 0$ YES!

So (-2) is root of both $P'(z)$ and $P(z)$, hence is the root of multiplicity 2. $\quad P(z) = (z+2)^2\, Q(z)$.

$z^4 + 4z^3 + 5z^2 + 4z + 4 = (z^2 + 4z + 4)(z^2 + 1)$

So 3 solutions: $z = -2$ (double solution), $z = i$, $z = -i$

18 If z is a complex number, solve $z^4 - 2z^2 + 9 = 0$, given that $1 + 2\sqrt{2}i = (\sqrt{2} + i)^2$.

Change of variable $X = z^2$

so $X^2 - 2X + 9 = 0$

$\Delta = 4 - 4 \times 9 = -32$

$\Delta = (4\sqrt{2}i)^2$

$$X = \frac{2 \pm 4\sqrt{2}i}{2} = 1 \pm 2\sqrt{2}i$$

$X_1 = 1 + 2\sqrt{2}i = z_1^2 = (\sqrt{2}+i)^2$ so $z_1 = \pm(\sqrt{2}+i)$

$X_2 = 1 - 2\sqrt{2}i = z_2^2 = (\sqrt{2}-i)^2 = 2 - 2\sqrt{2}i - 1 = 1 - 2\sqrt{2}i$

so $z_2 = \pm(\sqrt{2}-i)$

four roots $z = \sqrt{2} + i$

$z = -\sqrt{2} - i$

$z = \sqrt{2} - i$

$z = -\sqrt{2} + i$