THE PRODUCT RULE

1 Use the product rule to find the derivative of each function.

(a) $y = (x-2)(6x+7)$ (b) $f(x) = (2x+1)(x+3)$ (c) $y = (3x+4)(x^2-2x)$

(d) $g(x) = (x-1)(x^2-3x)$ (e) $y = (2x^2-5x)(x-2)$ (f) $f(x) = (x^2-4x)(x^2+3)$

a) $y = u(x) \times v(x)$ with $u(x) = x-2$ and $v(x) = 6x+7$

so $\frac{dy}{dx} = u'(x)\,v(x) + u(x)\,v'(x)$ $\quad u'(x) = 1$ $\quad v'(x) = 6$

$\frac{dy}{dx} = 1 \times (6x+7) + (x-2) \times 6 = 6x+7+6x-12 = 12x-5$

b) $u(x) = 2x+1$ $\quad v(x) = x+3$ $\quad \frac{dy}{dx} = 2(x+3) + 1 \times (2x+1) = 4x+7$

$u'(x) = 2$ $\quad v'(x) = 1$

c) $u(x) = 3x+4$ $\quad v(x) = x^2-2x$

$u'(x) = 3$ $\quad v'(x) = 2x-2$

$\frac{dy}{dx} = 3(x^2-2x) + (2x-2)(3x+4) = 9x^2-6x+8x-6x-8 = 9x^2-4x-8$

d) $u(x) = x-1$ $\quad v(x) = x^2-3x$

$u'(x) = 1$ $\quad v'(x) = 2x-3$

$\frac{dy}{dx} = 1 \times (x^2-3x) + (2x-3)(x-1) = 3x^2-3x-2x-3x+3 = 3x^2-8x+3$

e) $u(x) = 2x^2-5x$ $\quad v(x) = x-2$

$u'(x) = 4x-5$ $\quad v'(x) = 1$

$\frac{dy}{dx} = (4x-5)(x-2) + 1(2x^2-5x) = 6x^2-8x-5x-5x+10 = 6x^2-18x+10$

f) $u(x) = x^2-4x$ $\quad v(x) = x^2+3$

$u'(x) = 2x-4$ $\quad v'(x) = 2x$

$\frac{dy}{dx} = (2x-4)(x^2+3) + 2x(x^2-4x)$

$\frac{dy}{dx} = 2x^3-4x^2+6x-12+2x^3-8x^2$

$\frac{dy}{dx} = 4x^3-12x^2+6x-12$

THE PRODUCT RULE

4 For $g(x) = (x^2 + 5x)(x^3 + x^2 + 1)$, find: **(a)** $g'(x)$ **(b)** $g'(1)$ **(c)** $g'(-2)$

a) $g(x) = u(x) \times v(x)$ with $u(x) = x^2 + 5x$ $\quad v(x) = x^3 + x^2 + 1$

$u'(x) = 2x + 5 \quad v'(x) = 3x^2 + 2x$

$g'(x) = u'(x)\, v(x) + v'(x)\, u(x)$

$g'(x) = [2x+5][x^3 + x^2 + 1] + [3x^2 + 2x][x^2 + 5x]$

$g'(x) = 2x^4 + 2x^3 + 2x + 5x^3 + 5x^2 + 5 + 3x^4 + 15x^3 + 2x^3 + 10x^2$

$g'(x) = 5x^4 + 24x^3 + 15x^2 + 2x + 5$

b) $g'(1) = 5 \times 1^4 + 24 \times 1^3 + 15 \times 1^2 + 2 \times 1 + 5$

$g'(1) = 5 + 24 + 15 + 2 + 5 = 51$

c) $g'(-2) = 5 \times (-2)^4 + 24 \times (-2)^3 + 15 \times (-2)^2 + 2 \times (-2) + 5$

$g'(-2) = 5 \times 16 - 24 \times 8 + 15 \times 4 - 4 + 5$

$g'(-2) = 80 - 192 + 60 - 4 + 5$

$g'(-2) = -51$

THE PRODUCT RULE

5 Find $\frac{dy}{dx}$ for: (a) $y=\sqrt{x}(x-1)$ (b) $y=x\left(\sqrt{x}-1\right)$ (c) $y=x\left(\frac{1}{x}+1\right)$

a) $u(x)=\sqrt{x}=x^{1/2}$ $\quad u'(x)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-1/2}=\frac{1}{2x^{1/2}}=\frac{1}{2\sqrt{x}}$

$v(x)=x-1$ $\quad v'(x)=1$

So $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}(x-1)+1\times\sqrt{x}=\frac{x-1}{2\sqrt{x}}+\sqrt{x}=\frac{x-1+2x}{2\sqrt{x}}=\frac{3x-1}{2\sqrt{x}}$

b) $u(x)=x$ $\quad u'(x)=1$

$v(x)=\sqrt{x}-1$ $\quad v'(x)=\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx}=1\times(\sqrt{x}-1)+\frac{1}{2\sqrt{x}}\times x=\sqrt{x}-1+\frac{\sqrt{x}}{2}=\frac{3\sqrt{x}-2}{2}$

OR - Alternative method, by expanding first

$y=x\sqrt{x}-x=x^{3/2}-x$ $\quad$ so $\frac{dy}{dx}=\frac{3}{2}x^{\frac{3}{2}-1}-1=\frac{3}{2}x^{1/2}-1=\frac{3\sqrt{x}-2}{2}$

c) $u(x)=x$ $\quad u'(x)=1$

$v(x)=\frac{1}{x}+1=x^{-1}+1$ $\quad$ so $v'(x)=-x^{-1-1}=-x^{-2}=-\frac{1}{x^2}$

$\frac{dy}{dx}=1\times\left(\frac{1}{x}+1\right)+\left(-\frac{1}{x^2}\right)\times x=\frac{1}{x}+1-\frac{1}{x}=1$

OR - Alternative method, by expanding first

$y=1+x$ $\quad$ so $\frac{dy}{dx}=1$

THE PRODUCT RULE

5 Find $\frac{dy}{dx}$ for: **(d)** $y=(3\sqrt{x}+1)(x^2+4)$ **(e)** $y=\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$ **(f)** $y=\left(x^2+\frac{2}{x}\right)\left(1+\sqrt{x}\right)$

d) $u(x) = 3\sqrt{x}+1$ $\qquad u'(x) = \dfrac{3}{2\sqrt{x}}$

$v(x) = x^2+4$ $\qquad v'(x) = 2x$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x}}(x^2+4) + 2x(3\sqrt{x}+1) = \frac{3x^2+12}{2\sqrt{x}} + 6x\sqrt{x} + 2x$$

$$\frac{dy}{dx} = \frac{3x^2+12+12x^2+4x\sqrt{x}}{2\sqrt{x}} = \frac{15x^2+4x\sqrt{x}+12}{2\sqrt{x}} = \frac{15}{2}x\sqrt{x} + 2x + \frac{6}{\sqrt{x}}$$

OR - by expanding first: $y = 3x^2\sqrt{x} + 12\sqrt{x} + x^2 + 4 = 3x^{\frac{5}{2}} + 12\sqrt{x} + x^2$

$$\frac{dy}{dx} = \frac{15}{2}x^{3/2} + \frac{12}{2\sqrt{x}} + 2x = \frac{15}{2}x\sqrt{x} + \frac{6}{\sqrt{x}} + 2x$$

e) Expanding first $\quad y = x^2 - 1 + 1 - \dfrac{1}{x^2} = x^2 - \dfrac{1}{x^2} = x^2 - x^{-2}$

$$\frac{dy}{dx} = 2x - (-2)x^{-2-1} = 2x + 2x^{-3} = 2x + \frac{2}{x^3}$$

f) Expanding first $\quad y = x^2 + x^2\sqrt{x} + \dfrac{2}{x} + \dfrac{2}{\sqrt{x}}$

So $y = x^2 + x^{5/2} + 2x^{-1} + 2x^{-1/2}$

$$\frac{dy}{dx} = 2x + \frac{5}{2}x^{\frac{5}{2}-1} + 2\times(-1)x^{-1-1} + 2\times\left(\frac{-1}{2}\right)x^{-\frac{1}{2}-1}$$

$$\frac{dy}{dx} = 2x + \frac{5}{2}x^{3/2} - 2x^{-2} - x^{-3/2}$$

$$\frac{dy}{dx} = 2x + \frac{5}{2}x\sqrt{x} - \frac{2}{x^2} - \frac{1}{x\sqrt{x}}$$