RESISTED MOTION

2 A particle has an initial velocity U. After travelling a distance d in time T along a straight horizontal path, its velocity is V. The retardation of the particle at any time is proportional to its velocity at that time.

Show that: **(a)** $V = U - kd$ **(b)** $U = Ve^{kT}$ **(c)** $U = Ve^{\frac{T(U-V)}{d}}$

$m\ddot{x} = -km\dot{x}$ so $\ddot{x} = -k\dot{x}$ so $v\dfrac{dv}{dx} = -kv$

So $\dfrac{dv}{dx} = -k \iff dv = -k\,dx \Rightarrow \int_U^V dv = -k\int_0^d dx$

$\therefore$ $V - U = -kd$ $\therefore$ $V = U - kd$.

b) From $\ddot{x} = -k\dot{x} \Rightarrow \dfrac{\ddot{x}}{\dot{x}} = -k \Rightarrow \int_U^V \dfrac{\ddot{x}}{\dot{x}}\,dv = -k\int_0^T dt$

So $\left[\ln|\dot{x}|\right]_U^V = -kT$.

$\ln|V| - \ln|U| = -kT$

$\iff \ln\left(\dfrac{V}{U}\right) = -kT$

$\iff \dfrac{V}{U} = e^{-kT}$ so $V = Ue^{-kT}$

c) as $\ln\left(\dfrac{V}{U}\right) = -kT$ then $-k = \dfrac{1}{T}\ln\left(\dfrac{V}{U}\right)$

So the equation at a) becomes: $V = U + \dfrac{d}{T}\ln\left(\dfrac{V}{U}\right)$

$\iff \dfrac{(V-U)T}{d} = \ln\left(\dfrac{V}{U}\right) \iff \ln\left(\dfrac{U}{V}\right) = \dfrac{(U-V)T}{d}$

So $\dfrac{U}{V} = e^{\frac{T(U-V)}{d}}$ and $\therefore$ $U = V\exp\left[\dfrac{T(U-V)}{d}\right]$

RESISTED MOTION

4 An object of mass m falls from rest under constant gravitational force and against air resistance equal to kv, where v is the speed and k is a positive constant.

(a) Find its velocity at any time t.

(b) Sketch the velocity–time graph.

(c) Find the terminal velocity. Find the time taken to reach a speed v_1 where v_1 is one-quarter of the terminal velocity.

(d) Find the distance travelled when the speed v_1 is reached.

a) $m\ddot{x} = mg - kv$ so $\ddot{x} = g - \frac{k}{m}v$ so $\frac{dv}{dt} = g - \frac{k}{m}v$

$\therefore \frac{dv}{dt} = \frac{mg - kv}{m}$ so $\frac{dv}{mg - kv} = \frac{dt}{m}$

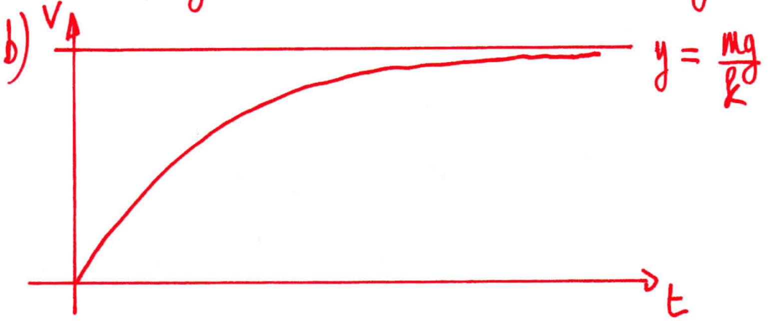
At $t=0$, $v=0$. We integrate both sides from $t=0$ to t.

$\int_0^v \frac{dv}{mg - kv} = \int_0^t \frac{dt}{m}$ so $-\frac{1}{k}\left[\ln|mg-kv|\right]_0^v = \frac{t}{m}$

$\iff \ln\left|\frac{mg-kv}{mg}\right| = -\frac{kt}{m} \iff \ln\left(1 - \frac{kv}{mg}\right) = -\frac{kt}{m}$

so $1 - \frac{kv}{mg} = e^{-kt/m} \Rightarrow \frac{kv}{mg} = 1 - e^{-kt/m}$ $v = \frac{mg}{k}\left[1 - e^{-kt/m}\right]$

b)



c) terminal velocity is

$\lim_{t\to+\infty} v = \lim_{t\to+\infty} \frac{mg}{k}\left[1 - e^{-kt/m}\right] = mg/k =$

let $v_1 = \frac{mg}{4k} \Rightarrow \frac{mg}{4k} = \frac{mg}{k}\left[1 - e^{-\frac{kt_1}{m}}\right] \Rightarrow \frac{1}{4} = 1 - e^{-kt_1/m}$

$\frac{k}{m} = \frac{g}{4v_1}$ so $e^{-\frac{gt_1}{4v_1}} = 3/4$ $\frac{gt_1}{4v_1} = \ln\left(\frac{4}{3}\right)$ so $t_1 = \frac{4v_1}{g}\ln\left(\frac{4}{3}\right)$

d) $v = V_{Ter}\left(1 - e^{-kt/m}\right)$ so $dx = V_{ter}\left(1 - e^{-\frac{kt}{m}}\right)dt$

$\int_0^x dx = \int_0^{t_1} V_{ter}\left(1 - e^{-kt/m}\right)dt$

So $x = V_{ter}\left[t - e^{-\frac{kt}{m}} \times \left(-\frac{m}{k}\right)\right]_0^{t_1}$

$$x = V_{ter}\left[\left(t_1 + e^{-\frac{kt_1}{m}} \times \frac{m}{k}\right) - \left(0 + \frac{m}{k}\right)\right]$$

$$x = V_{ter}\left[t_1 - \frac{m}{k} + \frac{m}{k} e^{-kt_1/m}\right]$$

But $V_{ter} = \frac{mg}{k}$ and $e^{-\frac{kt_1}{m}} = \frac{3}{4}$ so $t_1 = \frac{m}{k}\ln\left(\frac{4}{3}\right)$

$$x = \frac{mg}{k}\left[\frac{m}{k}\ln\left(\frac{4}{3}\right) - \frac{m}{k} + \frac{3m}{4k}\right]$$

$$x = \frac{m^2 g}{k^2}\left[\ln\left(\frac{4}{3}\right) - \frac{1}{4}\right]$$

RESISTED MOTION

6 A particle falls from rest under constant gravity and a resistance force. If the retardation due to the resistance force varies as the square of the velocity, find:

(a) the equation of motion (b) the terminal velocity
(c) the distance fallen as a function of the velocity
(d) the distance fallen when half the terminal velocity is reached
(e) the time taken to reach half the terminal velocity.

a) $m\ddot{x} = mg - mkv^2 \Rightarrow \ddot{x} = g - kv^2$

b) terminal velocity is reached when $\ddot{x} = 0$, i.e $V_T = \sqrt{\frac{g}{k}}$

c) $v\frac{dv}{dx} = g - kv^2$ so $\frac{v}{g - kv^2}\,dv = dx$

Integrating between $x = 0$ and x, and $v = 0$ and v, we obtain:

$$x = \int_{v=0}^{v} \frac{v}{g - kv^2}\,dv = \left[\frac{\ln|g - kv^2|}{-2k}\right]_0^v = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

d) At $v = \frac{1}{2}V_T$ $\quad v^2 = \frac{V_T^2}{4}$ so $v^2 = \frac{g}{4k}$ so $kv^2 = \frac{g}{4}$

$$x = \frac{1}{2k}\ln\left(\frac{g}{g - g/4}\right) = \frac{1}{2k}\ln\left(\frac{4}{3}\right)$$

e) $\ddot{x} = g - kv^2$ so $\frac{dv}{dt} = g - kv^2$

so $dt = \frac{dv}{g - kv^2}$ $\quad \therefore \int_0^t dt = \int_0^{V_T/2} \frac{dv}{g - kv^2} = \frac{1}{k}\int_0^{V_T/2} \frac{dv}{\frac{g}{k} - v^2}$

$$t = \frac{1}{k}\left[\int_0^{V_T/2} \frac{1}{\left(\sqrt{\frac{g}{k}} - v\right)} + \frac{1}{\left(\sqrt{\frac{g}{k}} + v\right)}\,dv\right] \times \sqrt{\frac{k}{g}} \times \frac{1}{2}$$

So $$t = \frac{1}{2\sqrt{kg}} \int_0^{V_T/2} \frac{1}{\left(\sqrt{\frac{g}{k}}+v\right)} + \frac{1}{\left(\sqrt{\frac{g}{k}}-v\right)} \, dv$$

But $V_T = \sqrt{\frac{g}{k}}$, so it simplifies as:

$$t = \frac{1}{2\sqrt{kg}} \int_0^{V_T/2} \left(\frac{1}{V_T+V} + \frac{1}{V_T-V} \right) dV$$

$$t = \frac{1}{2\sqrt{kg}} \Big[\ln(V_T+V) - \ln(V_T-V) \Big]_0^{V_T/2}$$

$$t = \frac{1}{2\sqrt{kg}} \left[\ln\left(\frac{V_T+V}{V_T-V} \right) \right]_0^{V_T/2}$$

$$t = \frac{1}{2\sqrt{gk}} \left(\ln\left(\frac{V_T+V_T/2}{V_T-V_T/2} \right) - \ln\left(\frac{V_T+0}{V_T-0} \right) \right)$$

$$t = \frac{1}{2\sqrt{gk}} \left(\ln\left(\frac{3/2}{1/2} \right) - \ln 1 \right)$$

$$t = \frac{1}{2\sqrt{gk}} \ln 3$$

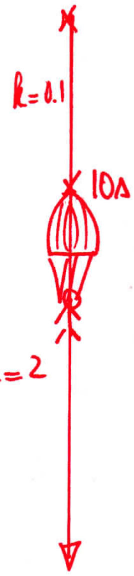
<u>Note</u>: The book solution say $t = \frac{1}{2\sqrt{g}} \ln 3$ but I suspect that's a mistake

RESISTED MOTION

9 A parachutist jumps from a stationary balloon at a great height. The parachute opens after 10 seconds. Assume that air resistance produces a retardation proportional to the velocity, with a constant of proportionality $k = 0.1$ for the first 10 seconds (i.e. during freefall) and $k = 2$ after the parachute opens. Find:

(a) the parachutist's velocity after 10 seconds **(b)** the parachutist's velocity after 15 seconds

(c) the parachutist's terminal velocity, i.e. the approximate velocity while floating to the ground.



a) $m\ddot{x} = mg - mkv$ so $\ddot{x} = g - kv$

$\frac{dv}{dt} = g - kv \Rightarrow dt = \frac{dv}{g - kv}$

$\int_0^t dt = \int_0^v \frac{dv}{g-kv}$ so $\left[-\frac{1}{k}\ln(g-kv)\right]_0^v = t$

$-kt = \ln(g-kv) - \ln(g)$ so $\ln\left(1 - \frac{kv}{g}\right) = -kt$

$1 - \frac{kv}{g} = e^{-kt}$ so $\frac{kv}{g} = 1 - e^{-kt}$ $\therefore v = \frac{g}{k}\left[1 - e^{-kt}\right]$

At $t = 10$s $v_{10} = \frac{g}{0.1}\left[1 - e^{-10\times0.1}\right] = 10g\left(1 - e^{-1}\right) = \frac{10g(e-1)}{e}$

b) $15 - 10 = \left[-\frac{1}{k}\ln(g-kv)\right]_{10}^{15}$ so $-5k = \ln\left(\frac{g - kv_{15}}{g - kv_{10}}\right)$

So $\frac{g - kv_{15}}{g - kv_{10}} = e^{-5\times2} = e^{-10}$

$g - kv_{15} = \frac{g - k\times\left(\frac{10g(e-1)}{e}\right)}{e^{10}}$ so $kv_{15} = g - \frac{g - \frac{20g(e-1)}{e}}{e^{10}}$

$v_{15} = \frac{g}{2}\left[\frac{e^{10} - 1 + 20 - 20/e}{e^{10}}\right] = \frac{g(e^{11} + 19e - 20)}{2e^{11}}$

c) $\ddot{x} = g - kv$. The terminal velocity is reached when $\ddot{x} = 0$,

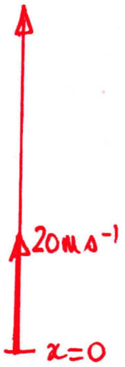
i.e. $g = kv_T$ or $v_T = \frac{g}{k} = \frac{g}{2}$

RESISTED MOTION

10 A particle is projected vertically upwards against air resistance. Its acceleration at any time t seconds after projection is given by $\ddot{x} = -\left(g + \frac{1}{10}v^2\right)$, where v m s^{-1} is the velocity. If the initial velocity is 20 m s^{-1}, find:

(a) the greatest height reached **(b)** the time taken to reach the greatest height.

a) $\ddot{x} = -\left(g + \frac{1}{10}v^2\right)$ so $v\frac{dv}{dx} = -\left(g + \frac{1}{10}v^2\right)$



$$\Rightarrow dx = \frac{v\,dv}{-\left(g + \frac{1}{10}v^2\right)}$$

$$x_{max} = \int_{20}^{0} \frac{v\,dv}{-\left(g + \frac{1}{10}v^2\right)} = \int_{0}^{20} \frac{v\,dv}{\left(g + \frac{1}{10}v^2\right)}$$

$$x_{max} = \left[\ln\left(g + \frac{1}{10}v^2\right) \times 5\right]_0^{20} = 5\left[\ln\left(g + \frac{400}{10}\right) - \ln(g)\right]$$

$$x_{max} = 5\ln\left(\frac{g+40}{g}\right) = 5\ln\left(1 + \frac{40}{g}\right)$$

b) $\ddot{x} = -\left(g + \frac{1}{10}v^2\right) = \frac{dv}{dt}$ so $-dt = \dfrac{dv}{g + \frac{1}{10}v^2}$

$$\int_0^t -dt = \int_{20}^{0} \frac{dV}{g + \frac{1}{10}V^2} = 10\int_{20}^{0} \frac{dV}{10g + V^2}$$

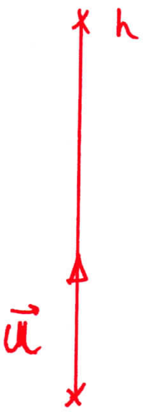
$$\int_t^0 dt = 10\left[\frac{1}{\sqrt{10g}}\tan^{-1}\left(\frac{V}{\sqrt{10g}}\right)\right]_{20}^{0} \quad \text{so} \quad t = \frac{10}{\sqrt{10g}}\left[\tan^{-1}\left(\frac{V}{\sqrt{10g}}\right)\right]_0^{20}$$

$$t = \sqrt{\frac{10}{g}}\left[\tan^{-1}\left(\frac{20}{\sqrt{10g}}\right) - \tan^{-1}0\right] = \sqrt{\frac{10}{g}}\tan^{-1}\left(2\sqrt{\frac{10}{g}}\right)$$

$$\therefore\ t = \sqrt{\frac{10}{g}}\tan^{-1}\left(2\sqrt{\frac{10}{g}}\right)$$

11 A particle is projected vertically upwards with initial speed u. Its acceleration is given by the differential equation $\ddot{x} = -(g + kv)$ where v is the speed at any time t, k is a positive constant and kv is the retardation due to air resistance.

(a) Find the maximum height reached by the particle.
(b) Find the time taken to reach the maximum height.
(c) Write the differential equation for the downward motion.
(d) Show that the particle returns to its point of projection with a speed V given by:
$k(u+V) = g\log_e\left[\dfrac{g+ku}{g-kV}\right]$



a) $\ddot{x} = -(g+kv) \Rightarrow v\dfrac{dv}{dx} = -(g+kv)$

so $\dfrac{-v\,dv}{g+kv} = dx$ we integrate both sides.

$$\int_0^h dx = \int_u^0 \frac{-v\,dv}{g+kv} = \int_0^u \frac{v\,dv}{g+kv}$$

let $y = g+kv$ so $\dfrac{dy}{dv} = k$

$v = \dfrac{y-g}{k}$ $dv = \dfrac{dy}{k}$

$$h - 0 = \int_g^{g+ku} \frac{\left(\frac{y-g}{k}\right)}{y}\frac{dy}{k} = \int_g^{g+ku} \frac{y-g}{yk^2}\,dy$$

$$h = \frac{1}{k^2}\int_g^{g+ku}\left(1 - \frac{g}{y}\right)dy = \frac{1}{k^2}\Big[y - g\ln|y|\Big]_g^{g+ku}$$

$$h = \frac{g+ku - g\ln|g+ku| - g + g\ln|g|}{k^2} = \frac{u}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+ku}\right)$$

b) $\ddot{x} = -(g+kv) \Rightarrow \dfrac{dv}{dt} = -(g+kv)$

So $dt = -\dfrac{dv}{g+kv}$ we integrate both sides.

$$\int_0^{t_{top}} dt = -\int_u^0 \frac{dv}{g+kv} = \int_0^u \frac{dv}{g+kv}$$

so $t_{top} - 0 = \left[\dfrac{\ln(g+kv)}{k}\right]_0^u = \dfrac{1}{k}\ln\left(\dfrac{g+ku}{g}\right) = \dfrac{1}{k}\ln\left(1+\dfrac{ku}{g}\right)$

 $\therefore\ t_{top} = \dfrac{1}{k}\ln\left(1+\dfrac{ku}{g}\right)$

c) $\ddot{x} = g - kv$

d) $\ddot{x} = g - kv \Rightarrow v\dfrac{dv}{dx} = g - kv \Rightarrow dx = \dfrac{v\,dv}{g-kv}$

So $\displaystyle\int_0^h dx = \int_0^{V_{h=0}} \frac{v\,dv}{g-kv}$ we do a change of variable

$y = g - kv \qquad \dfrac{dy}{dv} = -k \qquad dv = \dfrac{dy}{(-k)}$

$v = \dfrac{g-y}{k}$

$$h - 0 = \int_0^{V_{h=0}} \frac{\left(\frac{g-y}{k}\right)}{y} \times \left(-\frac{dy}{k}\right) = \frac{-1}{k^2}\int_g^{g-kV_{h=0}} \frac{g-y}{y}\,dy.$$

But $h = \dfrac{u}{k} + \dfrac{g}{k^2}\ln\left(\dfrac{g}{g+ku}\right)$ $= \dfrac{-1}{k^2}\displaystyle\int_g^{g-kV_{h=0}} \left(\frac{g}{y} - 1\right) dy$

$$h = \frac{-1}{k^2}\Big[g\ln(y) - y\Big]_g^{g-kV_{h=0}} = \frac{-1}{k^2}\left[g\ln\left(\frac{g-kV_{h=0}}{g}\right) - (g - kV_{h=0}) + g\right]$$

$h = \dfrac{-1}{k^2}\left[g\ln\left(\dfrac{g-kV}{g}\right) + kV\right]$, therefore:

$$\frac{u}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+ku}\right) = -\frac{V}{k} - \frac{g}{k^2}\ln\left(\frac{g-kV}{g}\right)$$

$\Leftrightarrow uk + Vk = g\left[-\ln\left(\dfrac{g-kV}{g}\right) - \ln\left(\dfrac{g}{g+ku}\right)\right]$

$\Leftrightarrow k(u+V) = g\left[\ln\left(\dfrac{g}{g-kV}\right) - \ln\left(\dfrac{g}{g+ku}\right)\right]$

$\Leftrightarrow k(u+V) = g\left[\ln g - \ln(g-kV) - \ln g + \ln(g+ku)\right]$

$\therefore k(u+V) = g\ln\left(\dfrac{g+ku}{g-kV}\right)$

14 A particle of unit mass moves in a horizontal straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(a) Show that: $\tan^{-1} Q - \tan^{-1} v = \tan^{-1}\left[\dfrac{Q-v}{1+Qv}\right]$

(b) Show that $x = \tan^{-1}\left[\dfrac{Q-v}{1+Qv}\right]$, where x is the displacement.

(c) Show that $t = \dfrac{1}{2}\log_e\left[\dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}\right]$, where t is the elapsed time when the particle is travelling with velocity v.

(d) Find v^2 as a function of t.

(e) Find the limiting values of v and x as $t \to \infty$.

$\times \xrightarrow{Q}$

$R = -(v+v^3)$

a) We take the tangent on both sides:

$$\tan\left(\tan^{-1}Q - \tan^{-1}v\right) = \frac{\tan(\tan^{-1}Q) - \tan(\tan^{-1}v)}{1+\tan(\tan^{-1}Q)\tan(\tan^{-1}v)} = \frac{Q-v}{1+Qv}$$

$$\therefore \quad \tan^{-1}Q - \tan^{-1}v = \tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$$

b) $\ddot{x} = -v - v^3$ so $v\dfrac{dv}{dx} = -v - v^3$

$$\Rightarrow \frac{dv}{dx} = -1 - v^2 \quad \Rightarrow \quad -dx = \frac{dv}{1+v^2}$$

$$\therefore \int_0^x -dx = \int_Q^v \frac{dv}{1+v^2} \quad \text{so } -x = \left[\tan^{-1}v\right]_Q^v$$

$$\text{So } x = \left[\tan^{-1}v\right]_v^Q = \tan^{-1}Q - \tan^{-1}v$$

And, using the result from a), $x = \tan^{-1}\left(\dfrac{Q-v}{1+Qv}\right)$

c) $\ddot{x} = -v - v^3$ but $\ddot{x} = \dfrac{dv}{dt}$ so $\dfrac{dv}{dt} = -v - v^3$

$$\therefore -dt = \frac{dv}{v+v^3} \quad \text{so } -\int_0^t dt = \int_Q^v \frac{dv}{v+v^3}$$

$$\frac{1}{v(1+v^2)} = \frac{a}{v} + \frac{bv+c}{1+v^2} = \frac{a+av^2+bv^2+cv}{v(1+v^2)}$$

so $a = 1$
$c = 0$
$a + b = 0$
so $b = -1$

$$-t = \int_Q^V \frac{1}{V} - \frac{V}{1+V^2}\, dV = \left[\ln V - \frac{\ln(1+V^2)}{2}\right]_Q^V$$

$$-t = \frac{1}{2}\left[2\ln V - \ln(1+V^2)\right]_Q^V = \frac{1}{2}\left[\ln V^2 - \ln(1+V^2)\right]_Q^V$$

$$-t = \frac{1}{2}\left[\ln\left(\frac{V^2}{1+V^2}\right)\right]_Q^V = \frac{1}{2}\left[\ln\left(\frac{V^2}{1+V^2}\right) - \ln\left(\frac{Q^2}{1+Q^2}\right)\right]$$

$$-t = \frac{1}{2}\left[\ln\left(\frac{V^2}{1+V^2}\right) + \ln\left(\frac{1+Q^2}{Q^2}\right)\right] = \frac{1}{2}\ln\left[\frac{V^2(1+Q^2)}{Q^2(1+V^2)}\right]$$

So indeed $t = \frac{1}{2}\ln\left(\frac{Q^2(1+V^2)}{V^2(1+Q^2)}\right)$

d) $\ln\left[\frac{Q^2(1+V^2)}{V^2(1+Q^2)}\right] = 2t \Rightarrow \frac{Q^2(1+V^2)}{V^2(1+Q^2)} = e^{2t}$

$$\Rightarrow Q^2 + Q^2V^2 = V^2\left[e^{2t}(1+Q^2)\right]$$

So $Q^2 = V^2\left[e^{2t}(1+Q^2) - Q^2\right]$ $\therefore V^2 = \frac{Q^2}{e^{2t}(1+Q^2) - Q^2}$

e) $\lim\limits_{t\to+\infty} V = \lim\limits_{t\to+\infty} \frac{Q}{\sqrt{\underbrace{e^{2t}(1+Q^2)-Q^2}_{}}} = 0$

tends towards $+\infty$

therefore, as $\lim\limits_{t\to+\infty} V = 0$

we look for $\lim\limits_{V\to 0} x = \lim\limits_{V\to 0} \tan^{-1}\left(\frac{Q-V}{1+QV}\right)$

$$\lim_{V\to 0} x = \tan^{-1}\left(\frac{Q-0}{1}\right) = \tan^{-1} Q$$

15 A particle of unit mass is projected vertically upwards in a medium in which the retardation due to resistance is $0.1v$. It is allowed to fall back to its point of projection. The initial speed of projection is V_0 and the final speed on return is V_F. Show that:

(a) the equation of motion on the upwards journey is $\ddot{x} = -(g + 0.1v)$

(b) the maximum height reached is $h = 10V_0 + 100g\log_e\left(\frac{10g}{10g + V_0}\right)$

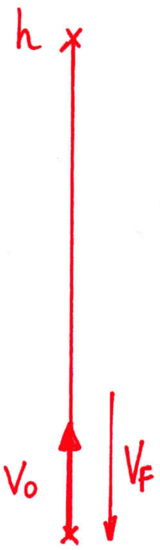
(c) the time taken to reach the highest point is $T_1 = 10\log_e\left(\frac{10g + V_0}{10g}\right)$

(d) the equation of motion on the downwards journey is $\ddot{x} = g - 0.1v$

(e) the time taken on the downwards journey is $T_2 = 10\log_e\left(\frac{10g}{10g - V_F}\right)$

(f) by analysis of the downwards journey, $h = -10V_F + 100g\log_e\left(\frac{10g}{10g - V_F}\right)$

(g) the total time of the motion is $T = \frac{V_0 + V_F}{g}$.



a) $m\ddot{x} = -mg - 0.1v \times m$ but $m = 1$

so $\ddot{x} = -(g + 0.1v)$

b) As $\ddot{x} = v\frac{dv}{dx}$, the equation becomes $v\frac{dv}{dx} = -(g + 0.1v)$

So $-dx = \frac{v\,dv}{g + 0.1v}$

$\therefore \int_0^h -dx = \int_{V_0}^{0} \frac{v}{g + 0.1v}\,dv$ let $y = g + 0.1v$ $\frac{dy}{dv} = 0.1$

$v = \frac{y - g}{0.1} = 10(y - g)$

So $\int_h^0 dx = \int_{g+0.1V_0}^{g} \frac{10(y-g)}{y} \times 10\,dy$

$\therefore -h = 100\int_{g+0.1V_0}^{g}\left(1 - \frac{g}{y}\right)dy = 100\left[y - g\ln(y)\right]_{g+0.1V_0}^{g}$

$\therefore -h = 100\left[(g - g\ln g) - (g + 0.1V_0 - g\ln(g + 0.1V_0))\right]$

$\therefore -h = 100\left[-0.1V_0 + g\ln\left(\frac{g + 0.1V_0}{g}\right)\right]$

$\therefore h = 10V_0 + 100g\ln\left(\frac{g}{g + 0.1V_0}\right)$

$\therefore h = 10V_0 + 100g\ln\left(\frac{10g}{10g + V_0}\right)$

c) as $\ddot{x} = \frac{dv}{dt}$ then $\frac{dv}{dt} = -(g + 0.1v)$

so $-dt = \frac{dv}{g + 0.1v}$ $\therefore -\int_0^{T_1} dt = \int_{V_0}^{0} \frac{dv}{g + 0.1v}$

$\therefore -T_1 = \left[\frac{\ln(g + 0.1v)}{0.1}\right]_{V_0}^{0} = 10\left[\ln(g + 0.1v)\right]_{V_0}^{0}$

So $T_1 = 10\left[\ln(g + 0.1v)\right]_0^{V_0} = 10\ln\left(\frac{g + 0.1V_0}{g}\right)$

$\therefore T_1 = 10\ln\left(\frac{10g + V_0}{10g}\right)$ d) $\ddot{x} = g - 0.1v$

e) $\frac{dv}{dt} = g - 0.1v \Rightarrow dt = \frac{dv}{g - 0.1v}$

$\int_0^{T_2} dt = \int_0^{V_F} \frac{dv}{g - 0.1v}$ so $T_2 = \left[\frac{\ln(g - 0.1v)}{(-0.1)}\right]_0^{V_F}$

$\therefore T_2 = 10\left[\ln(g - 0.1v)\right]_{V_F}^{0} = 10\ln\left(\frac{g}{g - 0.1V_F}\right) = 10\ln\left(\frac{10g}{10g - V_F}\right)$

f) $v\frac{dv}{dx} = g - 0.1v$, $\therefore dx = \frac{v\,dv}{g - 0.1v}$

$\int_0^h dx = \int_0^{V_F} \frac{v\,dv}{g - 0.1v}$ $y = g - 0.1v$, $v = (g - y)10$ $\frac{dy}{dv} = -0.1$ so $dv = -10\,dy$

$h = \int_g^{g - 0.1V_F} \frac{10(g - y)}{y} \times (-10\,dy) = 100\int_{g - 0.1V_F}^{g} \frac{g - y}{y}\,dy = 100\int_{g - 0.1V_F}^{g} \left(\frac{g}{y} - 1\right) dy$

$h = 100\left[g\ln y - y\right]_{g - 0.1V_F}^{g}$

$$h = 100\left[\left(g \ln g - g\right) - \left(g \ln\left(g - 0.1V_F\right) - \left(g - 0.1V_F\right)\right)\right]$$

$$h = 100\left[g \ln\left(\frac{g}{g - 0.1V_F}\right) - 0.1 V_F\right]$$

$$\therefore\ h = 100g \ln\left(\frac{g}{g - 0.1V_F}\right) - 10 V_F$$

$$h = -10V_F + 100g \ln\left(\frac{10g}{10g - V_F}\right)$$

g) We add the time to go up and the time to go down

$$T = T_1 + T_2 = 10 \ln\left(\frac{10g + V_0}{10g}\right) + 10 \ln\left(\frac{10g}{10g - V_F}\right)$$

$$T = 10 \ln\left[\frac{(10g + V_0)}{10g} \times \frac{10g}{(10g - V_F)}\right] = 10 \ln\left(\frac{10g + V_0}{10g - V_F}\right)$$

We can equate the heights found at b) and at f), $\therefore$

$$10V_0 + 100g \ln\left(\frac{10g}{10g + V_0}\right) = -10V_F + 100g \ln\left(\frac{10g}{10g - V_F}\right)$$

$$\therefore\ V_0 + V_F = 10g\left[\ln\left(\frac{10g}{10g - V_F}\right) - \ln\left(\frac{10g}{10g + V_0}\right)\right]$$

$$V_0 + V_F = 10g\left[\ln\left[\left(\frac{10g}{10g - V_F}\right)\left(\frac{10g + V_0}{10g}\right)\right]\right] = 10g \ln\left(\frac{10g + V_0}{10g - V_F}\right)$$

$$\therefore\ 10 \ln\left(\frac{10g + V_0}{10g - V_F}\right) = \frac{V_0 + V_F}{g}$$

Therefore $T = T_1 + T_2 = \dfrac{V_0 + V_F}{g}$