PARTIAL FRACTIONS, QUADRATIC FACTORS

2 Reduce each rational function to its partial fractions.

(a) $\dfrac{3}{(x^2+1)(x^2+4)}$ **(b)** $\dfrac{8}{(x^2+1)(x^2+9)}$

a) $\dfrac{3}{(x^2+1)(x^2+4)} = \dfrac{ax+b}{x^2+1} + \dfrac{cx+d}{x^2+4}$

$$= \frac{x^3(a+c)+x^2(b+d)+x(4a+c)+(4b+d)}{(x^2+1)(x^2+4)}$$

$$\therefore \begin{cases} a+c=0 \\ b+d=0 \\ 4a+c=0 \\ 4b+d=3 \end{cases} \Leftrightarrow \begin{cases} c=-a \\ 4a-a=0 \\ b+d=0 \\ 4b+d=3 \end{cases}$$

so $a=0$ and $c=0$

so $3b=3$ $\boxed{b=1}$

and $d=-b=-1$

$$\frac{3}{(x^2+1)(x^2+4)} = \frac{1}{x^2+1} - \frac{1}{x^2+4}$$

b) $\dfrac{8}{(x^2+1)(x^2+9)} = \dfrac{ax+b}{x^2+1} + \dfrac{cx+d}{x^2+9}$

$$= \frac{x^3(a+c)+x^2(b+d)+x(9a+c)+(d+9b)}{(x^2+1)(x^2+9)}$$

$$\begin{cases} a+c=0 \\ b+d=0 \\ 9a+c=0 \\ d+9b=8 \end{cases} \Leftrightarrow \begin{cases} c=-a \\ d=-b \\ 9a-a=0 \\ -b+9b=8 \end{cases}$$

so $a=0$ and $c=0$

so $b=1$ and $d=-1$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{1}{x^2+1} - \frac{1}{x^2+9}$$

PARTIAL FRACTIONS, QUADRATIC FACTORS

(a) $\dfrac{x^2-8x+2}{(x-2)(x^2+1)}$ (b) $\dfrac{x^2+4x+1}{(x+2)(x^2+x+1)}$

a) $\dfrac{x^2-8x+2}{(x-2)(x^2+1)} = \dfrac{a}{x-2} + \dfrac{bx+c}{x^2+1} = \dfrac{x^2(a+b)+x(c-2b)+(a-2c)}{(x-2)(x^2+1)}$

$\begin{cases} a+b=1 \\ c-2b=-8 \\ a-2c=2 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ c-2(1-a)=-8 \\ a-2c=2 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ 2a+c=-6 \\ a-2c=2 \end{cases}$ $\quad c=-6-2a$

So $a-2(-6-2a)=2 \Leftrightarrow 5a=2-12=-10$ $\boxed{a=-2}$

$\therefore c=-6-2\times(-2)=-2$ and $b=1-(-2)=3$

$\dfrac{x^2-8x+2}{(x-2)(x^2+1)} = \dfrac{-2}{x-2} + \dfrac{3x-2}{x^2+1}$

b) $\dfrac{x^2+4x+1}{(x+2)(x^2+x+1)} = \dfrac{a}{x+2} + \dfrac{bx+c}{x^2+x+1} = \dfrac{x^2(a+b)+x(a+2b+c)+(2c+a)}{(x+2)(x^2+x+1)}$

$\begin{cases} a+b=1 \\ a+2b+c=4 \\ 2c+a=1 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ a+2(1-a)+c=4 \\ 2c+a=1 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ -a+c=2 \\ 2c+a=1 \end{cases}$

So $3c=3$ $\boxed{c=1}$ then $a=c-2=1-2=-1$

$\boxed{a=-1}$

and $b=1-a=1-(-1)=2$ $\boxed{b=2}$

$\dfrac{x^2+4x+1}{(x+2)(x^2+x+1)} = \dfrac{-1}{x+2} + \dfrac{2x+1}{x^2+x+1}$

(a) $\dfrac{x^2+9}{(x+2)(x^2-2x+5)} = \dfrac{a}{x+2} + \dfrac{bx+c}{x^2-2x+5} = \dfrac{x^2(a+b)+x(-2a+2b+c)+5a+2c}{(x+2)(x^2-2x+5)}$

$\Delta = 2^2 - 4\times 5 = -16 < 0$

$$\begin{cases} a+b=1 \\ -2a+2b+c=0 \\ 5a+2c=9 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ -2a+2(1-a)+c=0 \\ 5a+2c=9 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ -4a+c=-2 \\ 5a+2c=9 \end{cases}$$

$$\Leftrightarrow \begin{cases} b=1-a \\ 8a-2c=4 \\ 5a+2c=9 \end{cases}$$

So $13a = 13$ $\boxed{a=1}$

$b = 1-a = 0$ $\boxed{b=0}$

and $c = -2+4a = -2+4 = 2$

$\boxed{c=2}$

$$\therefore \frac{x^2+9}{(x+2)(x^2-2x+5)} = \frac{1}{x+2} + \frac{2}{x^2-2x+5}$$

(d) $\dfrac{x^4+3x^2-8x}{x^3-8}$

$x^3-8=(x-2)(x^2+2x+4)$

for the quadratic, $\Delta=2^2-4\times4<0$ so no factorisable.

$$\therefore \frac{x^4+3x^2-8x}{x^3-8}=\frac{x(x^3-8)+3x^2}{x^3-8}=x+\frac{3x^2}{x^3-8}$$

For the 2nd term:

$$\frac{3x^2}{x^3-8}=\frac{a}{x-2}+\frac{bx+c}{x^2+2x+4}$$

$$=\frac{x^2(a+b)+x(2a-2b+c)+(4a-2c)}{x^3-8}$$

$$\therefore \begin{cases} a+b=3 \\ 2a-2b+c=0 \\ 4a-2c=0 \end{cases} \Leftrightarrow \begin{cases} b=3-a \\ c=2a \\ 2a-2(3-a)+2a=0 \end{cases}$$

So $6a-6=0$ $\boxed{a=1}$

so $b=3-1=2$ $\boxed{b=2}$ and $\boxed{c=2}$

$$\frac{x^4+3x^2-8x}{x^3-8}=x+\frac{1}{x-2}+\frac{2x+2}{x^2+2x+4}$$

(e) $\dfrac{x^2+1}{x^3-1} = \dfrac{x^2+1}{(x-1)(x^2+x+1)}$ $\quad \Delta = 1-4 = -3 < 0$

$$\frac{\quad}{} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$$

$$\frac{\quad}{} = \frac{x^2(a+b) + x(a+c-b) + (a-c)}{x^3-1}$$

$$\therefore \begin{cases} a+b=1 \\ a-b+c=0 \\ a-c=1 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ c=a-1 \\ a-(1-a)+(a-1)=0 \end{cases}$$

$$\Leftrightarrow 3a-2=0 \qquad \boxed{a=2/3}$$

$$b = 1-\frac{2}{3} = \frac{1}{3} \qquad \boxed{b=1/3} \qquad c = a-1 = \frac{2}{3}-1 \qquad \boxed{c=-1/3}$$

$$\frac{x^2+1}{x^3-1} = \frac{2/3}{x-1} + \frac{1/3\,x - 1/3}{x^2+x+1}$$

$$\frac{\quad}{} = \frac{2}{3(x-1)} + \frac{x-1}{3(x^2+x+1)}$$

(f) $\dfrac{x+3}{(2x+1)(x^2+1)} = \dfrac{a}{2x+1} + \dfrac{bx+c}{x^2+1}$

$$= \frac{x^2(a+2b)+x(b+2c)+(a+c)}{(2x+1)(x^2+1)}$$

$$\therefore \begin{cases} a+2b=0 \\ b+2c=1 \\ a+c=3 \end{cases} \Leftrightarrow \begin{cases} a=-2b \\ -2b+c=3 \\ b+2c=1 \end{cases} \Leftrightarrow \begin{cases} a=-2b \\ b=1-2c \end{cases}$$

so $-2(1-2c)+c=3$ $\quad 4c-2=3$

so $5c=5$ $\quad \boxed{c=1}$

$b=1-2\times 1=-1$ $\quad \boxed{b=-1}$

and $a=-2\times(-1)=2$ $\quad \boxed{a=2}$

$$\therefore \frac{x+3}{(2x+1)(x^2+1)} = \frac{2}{2x+1} + \frac{-x+1}{x^2+1}$$