RELATED RATES OF CHANGE

1 If $A = \pi r^2$ and $C = 2\pi r$, the expression for $\frac{dA}{dC}$ is given by:

A 2π **B** $\frac{1}{2\pi}$ $\boxed{\textbf{C}}$ r **D** $\frac{1}{r}$

$$\frac{dA}{dC} = \frac{dA}{dr} \times \frac{dr}{dC} = 2\pi r \times \frac{1}{\frac{dC}{dr}} = 2\pi r \times \frac{1}{2\pi} = r \text{ so } \boxed{C}$$

2 If $x^2 + y^2 = 144$, find the value of $\frac{dy}{dt}$ when $\frac{dx}{dt} = 0.6$ and $x = 5$, given that x and y are both positive.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \qquad y^2 = 144 - x^2 \text{ so } y = \sqrt{144 - x^2} = (144 - x^2)^{1/2}$$

$$\text{So } \frac{dy}{dx} = \frac{1}{2}(144 - x^2)^{1/2 - 1} \times (-2x) = \frac{-x}{(144 - x^2)^{-1/2}} = \frac{-x}{\sqrt{144 - x^2}}$$

$\therefore$ at $x = 5$ and when $\frac{dx}{dt} = 0.6$, we obtain:

$$\frac{dy}{dt} = \left(\frac{-5}{\sqrt{144 - 5^2}}\right) \times 0.6 = \frac{-3}{\sqrt{119}}$$

5 Given $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{h}{4}$, answer the following questions.

(a) Find the expression for V in terms of h. **(b)** If $\frac{dV}{dt} = 0.6$, find the expression for $\frac{dh}{dt}$

(c) Find the value of $\frac{dh}{dt}$ when $r = 1$.

a) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 \times h = \frac{\pi h^3}{48}$

b) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt} = \frac{1}{\frac{3\pi h^2}{48}} \times 0.6 = \frac{9.6}{\pi h^2}$

c) when $r = 1$, $h = 4$, so

$$\frac{dh}{dt} = \frac{9.6}{\pi \times 4^2} = \frac{3}{5\pi}$$

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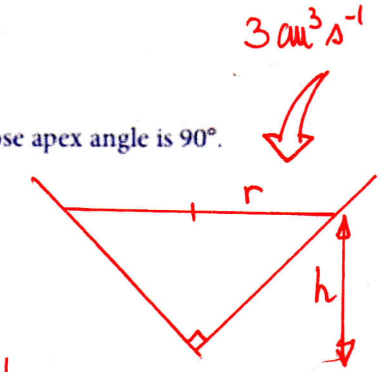
8 Water is being poured at a constant rate of 3 cm^3/s into an inverted right conical vessel whose apex angle is 90°. At what rate is the water level rising when the depth is π cm?



$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} \quad \text{with } \frac{dV}{dt} = 3\text{cm}^3\text{s}^{-1}$$

$$\text{So } \frac{dh}{dt} = \frac{1}{\frac{dV}{dh}} \times 3$$

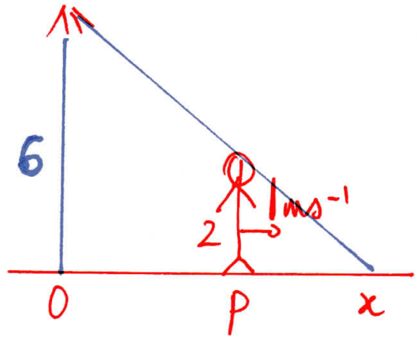
$V = \frac{1}{3}\pi r^2 h$ but $r = h$ as the angle of the cone is 90°.

so $V = \frac{\pi h^3}{3}$

$$\frac{dh}{dt} = \frac{1}{\frac{\pi}{\cancel{3}} \times \cancel{3}h^2} \times 3 = \frac{3}{\pi h^2}$$

when $h = \pi$ cm, $\quad \frac{dh}{dt} = \frac{3}{\pi \times \pi^2} = \frac{3}{\pi^3}$ cm s^{-1}

9 A lamp is 6 m directly above a straight footpath. A person 2 m tall walks along the footpath away from the light at a constant speed of 1 m/s. At what speed is the end of the person's shadow moving along the path? At what speed is the length of the shadow increasing?



$$\frac{dx}{dt} = \frac{dx}{dp} \times \frac{dp}{dt} \qquad \frac{dp}{dt} = 1\text{ms}^{-1}$$

The triangles are similar, $\therefore \frac{6}{x} = \frac{2}{x-p}$

$$\Leftrightarrow \frac{3}{x} = \frac{1}{x-p} \Leftrightarrow 3(x-p) = x$$

$$\Leftrightarrow 2x = 3p \Leftrightarrow x = \frac{3}{2}p.$$

$\therefore \frac{dx}{dp} = \frac{3}{2} \times 1 = \frac{3}{2}$ ms^{-1} (speed at the end of the person's shadow)

$\frac{d(x-p)}{dt} = \frac{dx}{dt} - \frac{dp}{dt} = \frac{3}{2} - 1 = 0.5$ ms^{-1} (speed at which the length of the shadow is increasing)

RELATED RATES OF CHANGE

so $\frac{dV}{dt} = 8\ cm^3/min$

12 A melting snowball is decreasing in volume at a constant rate of 8 cm^3/min. If the melting snowball is always a perfect spherical shape, find the rate at which its radius is changing when the radius is 4 cm.

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} \qquad \text{But} \quad V = \frac{4}{3}\pi r^3$$

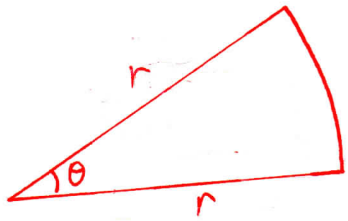
$$\frac{dr}{dt} = \frac{1}{\frac{dV}{dr}} \times 8 = \frac{1}{\frac{4}{\cancel{3}}\pi \times \cancel{3}r^2} \times 8 = \frac{2}{\pi r^2}$$

when $r = 4$ cm, $\quad \frac{dr}{dt} = \frac{2}{\pi \times 4^2} = \frac{1}{8\pi}$ cm/min

14 The perimeter of a circular sector is 20 cm. The radius is increasing at a rate of 5 cm/s.

(a) At what rate is the angle of the sector changing when the radius length is 10 cm?

(b) At what rate is the area changing when the radius is 10 cm?



a) $P = 2r + r\theta = r(\theta + 2)$.

so $\theta = \frac{P}{r} - 2 = \frac{20}{r} - 2$. as $P = 20$ cm.

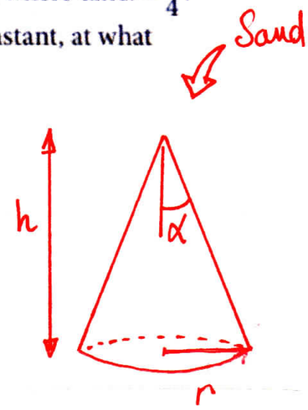
$$\therefore \frac{d\theta}{dt} = \frac{d\theta}{dr} \times \frac{dr}{dt} = -\frac{20}{r^2} \times 5 = -\frac{20}{10^2} \times 5 = -1 \text{ rad } s^{-1}$$

b) $\text{Area} = \frac{1}{2} r^2 \theta$

$$\text{so } \frac{dA}{dt} = \frac{r^2}{2} \times \frac{d\theta}{dt} = \frac{10^2}{2} \times (-1) = -50 \text{ cm}^2 s^{-1}$$

RELATED RATES OF CHANGE

16 Sand is poured into a heap in the shape of a right circular cone whose semi-vertex angle is α, where $\tan\alpha = \frac{3}{4}$. When the height of the cone is 16 cm, the height is increasing at a rate of 2 cm/min. At that instant, at what rate is the volume increasing?



$V = \frac{1}{3}\pi r^2 \times h$ But $\tan\alpha = \frac{r}{h} \Leftrightarrow r = h\tan\alpha$

$r = \frac{3h}{4}$

So $V = \frac{1}{3}\pi\left(\frac{3h}{4}\right)^2 \times h = \frac{3\pi h^3}{16}$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{3\pi}{16} \times 3h^2 \times 2$$

$$\frac{dV}{dt} = \frac{9\pi}{8} \times 16^2 = 288\pi \text{ cm}^3/\text{min}$$

17 When a certain gas expands at constant temperature, its pressure P and volume V are given by the relation $PV^{1.4} = k$, a constant. At a certain instant the pressure is 25 g/cm^2 and the volume is 32 cm^3. If the volume is increasing at the rate of $5 \text{ cm}^3/\text{s}$, at what rate is the pressure changing at that instant?

we look for $\frac{dP}{dt}$ $\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$

But $P = \frac{k}{V^{1.4}} = kV^{-1.4}$ so $\frac{dP}{dV} = -1.4kV^{-1.4-1} = -1.4kV^{-2.4}$

When $P = 25 \text{ g/cm}^3$ and $V = 32 \text{ cm}^3$, $PV^{1.4} = 25 \times 32^{1.4} = 3200$

so $k = 3200$.

$$\frac{dP}{dt} = -1.4 \times 3{,}200 \times 32^{-2.4} \times 5 = -5.46875 \text{ g cm}^{-2}\text{s}^{-1}$$

or $\frac{dP}{dt} = -5\frac{15}{32} \text{ g cm}^{-2}\text{s}^{-1}$

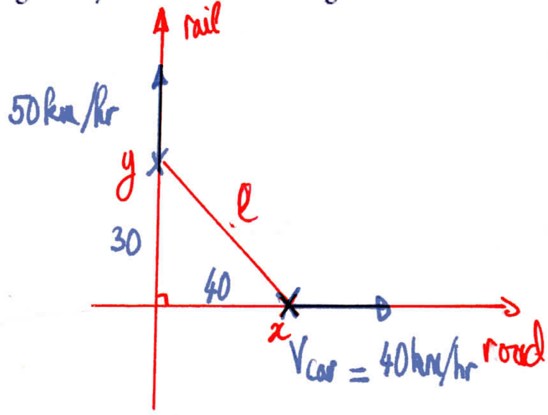
(Obviously the pressure is decreasing as the overall volume is increasing)

RELATED RATES OF CHANGE

21 A straight railway track and a straight road intersect at right angles. At a given instant a car travelling at 40 km/h and a train travelling at 50 km/h are moving away from the intersection and are 40 km and 30 km from the intersection respectively.

(a) The car and train continue moving in straight lines without changing their speed. One hour later, at what rate is the distance between the car and the train changing?

(b) At what rate would the distance between the car and train be changing if they were both travelling towards the intersection?



a) $\frac{dx}{dt} = 40$ and $\frac{dy}{dt} = 50$

we look for $\frac{dl}{dt}$.

$l^2 = x^2 + y^2$ so $l = \sqrt{x^2+y^2}$

$x = 40 + 40t$ and $y = 30 + 50t$, so

$l = \sqrt{(40+40t)^2 + (30+50t)^2} = \sqrt{4100t^2 + 6200t + 2500}$

So $l = 10\sqrt{41t^2+62t+25} = 10(41t^2+62t+25)^{1/2}$

So $\frac{dl}{dt} = \frac{10}{2}(41t^2+62t+25)^{1/2-1} \times (2\times 41t + 62)$.

So $\frac{dl}{dt} = \frac{5(82t+62)}{\sqrt{41t^2+62t+25}}$. At $t=1$, $\frac{dl}{dt} = \frac{5(82\times1+62)}{\sqrt{41+62+25}} = \frac{90}{\sqrt{2}}$

So at $t=1$, $\frac{dl}{dt} = 45\sqrt{2}$ km hr^{-1}

b) if $x = 40 - 40t$ and $y = 30 - 50t$, then

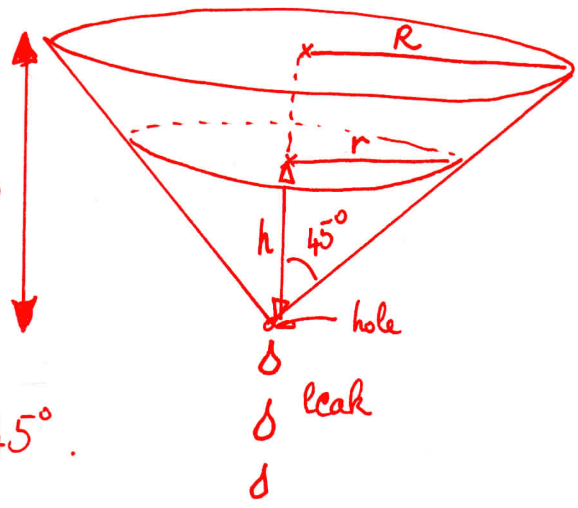
$l = \sqrt{(40-40t)^2 + (30-50t)^2} = \sqrt{4100t^2 - 6200t + 2500} = 10(41t^2 - 62t + 25)^{1/2}$

$\therefore \frac{dl}{dt} = \frac{10}{2}(41t^2 - 62t + 25)^{1/2} \times (2\times41t - 62) = \frac{5(82t-62)}{\sqrt{41t^2-62t+25}}$

$\therefore$ in that case $\frac{dl}{dt} = \frac{5(82\times1-62)}{\sqrt{41\times1^2-62\times1+25}} = \frac{100}{\sqrt{4}} = 50$ km hr^{-1}

RELATED RATES OF CHANGE

23 A conical tank with a vertical axis has a semi-vertical angle of 45°. Water, initially at a depth of 5 metres, leaks out through a hole at the bottom of the tank at a rate of $0.2\sqrt{h}$ m^3/min when the depth is h metres. Find the rate at which the depth is decreasing when the depth is 4 metres.



We look for $\frac{dh}{dt}$ (rate at which the water is decreasing).

the volume of water in the tank is:

$$V = \frac{1}{3}\pi r^2 \times h$$

but $h = r$ as the semi-vertical angle is 45°.

So $V = \frac{\pi h^3}{3}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt}$$

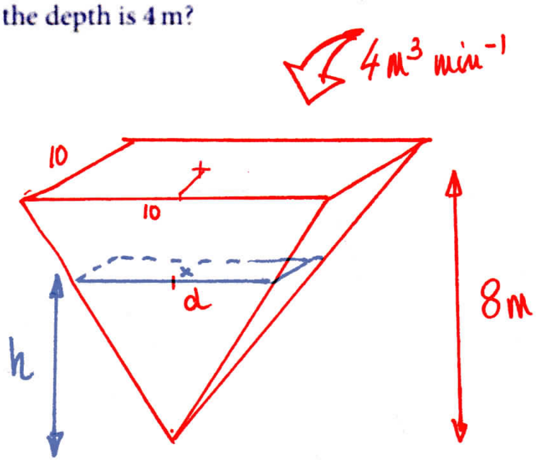
$$\therefore \frac{dh}{dt} = \frac{1}{\frac{3\pi h^2}{3}} \times 0.2\sqrt{h} = \frac{0.2\sqrt{h}}{\pi h^2} = \frac{0.2}{\pi h\sqrt{h}}$$

So when $h = 4$, $\frac{dh}{dt} = \frac{0.2}{\pi \times 4 \times \sqrt{4}} = \frac{0.2}{8\pi}$

$$\frac{dh}{dt} = \frac{1}{40\pi} \text{ m min}^{-1}$$

RELATED RATES OF CHANGE

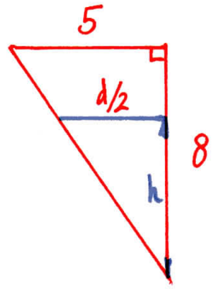
25 A loading chute is in the shape of a square pyramid with base length 10 m and depth 8 m. Liquid is poured in at the top at a rate of 4 m³/min. At what rate is the level rising when the depth is 4 m?



We look for $\frac{dh}{dt}$ (rate at which the level is rising)

$$V = \frac{1}{3} d^2 \times h$$

we need to find a relationship between d and h.



The two triangles are similar, $\therefore \ \frac{5}{8} = \frac{d/2}{h}$

or $5h = 4d. \iff d = \frac{5}{4}h.$

So $V = \frac{1}{3}\left(\frac{5}{4}h\right)^2 \times h = \frac{25h^3}{48}$

$$\therefore \ \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt} = \frac{1}{\frac{3 \times 25h^2}{48}} \times 4$$

$$\frac{dh}{dt} = \frac{64}{25h^2}$$

When $h = 4$, $\frac{dh}{dt} = \frac{64}{25 \times 4^2} = 0.16$ m min⁻¹

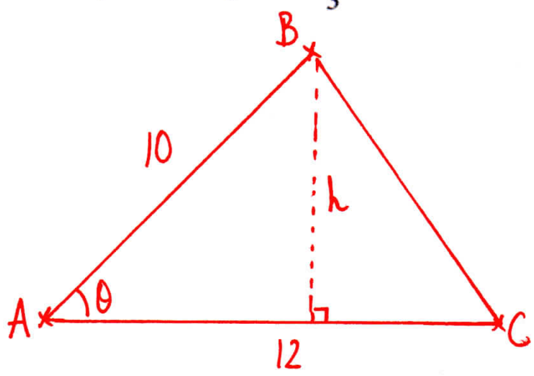
RELATED RATES OF CHANGE

$\frac{d\theta}{dt} = 0.1 \text{ rad s}^{-1}$

27 In triangle ABC, $AB = 10$ cm, $AC = 12$ cm and angle A is increasing at the rate of 0.1 radians per second. At what rate is:

(a) the area of ΔABC increasing

(b) the length of BC increasing, when angle A is $\frac{\pi}{3}$ radians?



a) $A = \text{Area} = \frac{h \times 12}{2} = 6h$

$\sin\theta = \frac{h}{10}$ so $h = 10\sin\theta$

$A = 60\sin\theta$

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt} = 60\cos\theta \times 0.1 = 6\cos\theta$

NOTE
$\frac{d(\sin\theta)}{d\theta} = \cos\theta$
$\frac{d(\cos\theta)}{d\theta} = -\sin\theta$

b) Cosine rule: $BC^2 = 10^2 + 12^2 - 2 \times 10 \times 12\cos\theta$

So $BC^2 = 244 - 240\cos\theta$ $\qquad BC = (244 - 240\cos\theta)^{1/2}$

$\frac{d(BC)}{dt} = \frac{d(BC)}{d\theta} \times \frac{d\theta}{dt} = \frac{1}{2}(244 - 240\cos\theta)^{1/2 - 1} \times [-240(-\sin\theta)] \times 0.1$

So $\frac{d(BC)}{dt} = \frac{12\sin\theta}{\sqrt{244 - 240\cos\theta}}$

when $\theta = \pi/3$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$

so $\frac{d(BC)}{dt} = \frac{12 \times \sqrt{3}/2}{\sqrt{244 - 240 \times 1/2}} = \frac{6\sqrt{3}}{\sqrt{124}} = \frac{3\sqrt{3}}{\sqrt{31}}$ cm s^{-1}

RELATED RATES OF CHANGE

28 A spherical mothball evaporates at a rate proportional to its surface area so that its volume $V\,\text{cm}^3$ and radius r after t weeks are related by the equation $\frac{dV}{dt} = -4k\pi r^2$, where k is a positive constant.

(a) Show that $\frac{dr}{dt} = -k$.

(b) If the initial radius of the mothball is 1 cm and the radius after 10 weeks is 0.5 cm, express r in terms of t.

a) For a sphere, $V = \frac{4}{3}\pi r^3$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dr}} \times (-4k\pi r^2) = \frac{-4k\cancel{\pi}\cancel{r^2}}{\cancel{3} \times \frac{4\cancel{\pi}\cancel{r^2}}{\cancel{3}}} = -k$$

So indeed $\frac{dr}{dt} = -k$ where k is a constant.

b) as $\frac{dr}{dt} = -k$, then $r(t) = -kt + \text{Constant}$. (by integration).

At $t=0$, $r=1$ so $1 = \text{Constant}$

so $r(t) = -kt + 1$

At $t = 10$, $r = 0.5$, so $0.5 = -k \times 10 + 1$

so $k = \frac{0.5}{10} = 0.05$

So $r(t) = 1 - 0.05t$

RELATED RATES OF CHANGE

30 **(a)** Show that the formula for the volume V of a right circular cone of base radius r and height h can be expressed as $V = \frac{1}{3}\pi h^3 \tan^2\alpha$, where α is the semi-vertex angle.

(b) Water flows out through a hole at the vertex angle of an inverted cone, whose angle is 60°, at a rate equal to π times the square root of the depth of the water at any time. At what rate (in cm/s) would the water level be dropping when the depth is 9 cm?

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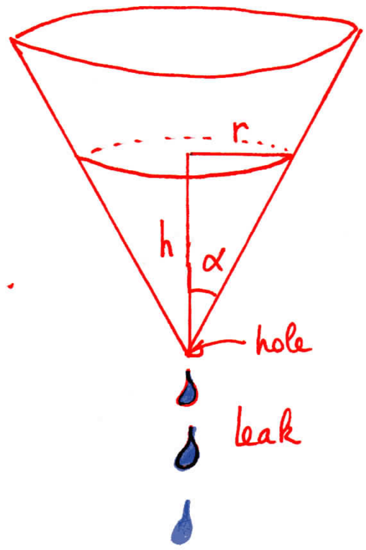
a) $V = \frac{1}{3}\pi r^2 \times h$

But $\tan\alpha = \frac{r}{h}$ so $r = h\tan\alpha$

So $V = \frac{1}{3}\pi (h\tan\alpha)^2 \times h = \frac{1}{3}\pi h^3 \tan^2\alpha.$



b) $\frac{dV}{dt} = \pi \times \sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times \pi\sqrt{h}$

$\therefore \frac{dh}{dt} = \frac{1}{\frac{1}{\cancel{3}}\cancel{\pi}\cancel{3}h^2\tan^2\alpha} \times \cancel{\pi}\sqrt{h} = \frac{1}{h\sqrt{h}\tan^2\alpha}$

$\alpha = 30°$, so $\tan\alpha = \frac{\sin 30}{\cos 30} = \frac{1}{\sqrt{3}}$ so $\tan^2\alpha = \frac{1}{3}$

$\frac{dh}{dt} = \frac{3}{9\sqrt{9}} = \frac{1}{9}$ cm s^{-1}