INEQUALITIES

1 Show that if $a \geq 0$ and $b \geq 0$ then $ab(a^2 + b^2) \geq 2a^2b^2$.

2 If $0 < x < y$, prove that $x^2 < xy < y^2$.

INEQUALITIES

3 **(a)** For positive x and y, prove that $\frac{x}{y}+\frac{y}{x}\geq 2$.

(b) Hence prove that $x^2-xy+y^2\geq xy$.

(c) Factorise x^3+y^3 and show that $x^3+y^3\geq xyz\left(\frac{x}{z}+\frac{y}{z}\right)$ for $x, y, z>0$.

(d) Write similar expressions for y^3+z^3 and z^3+x^3.

(e) Using results from parts **(c)** and **(d)**, prove that $x^3+y^3+z^3\geq 3xyz$.

(f) If a, b, c, d are positive, deduce that:

(i) $a+b+c\geq 3\sqrt[3]{abc}$

(ii) $(a+b+c)(a+b+d)(a+c+d)(b+c+d)\geq 81abcd$

INEQUALITIES

4 **(a)** Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for real a, b, c.

(b) Hence show that $(a + b + c)^2 \geq 3(ab + bc + ca)$.

6 **(a)** Prove that $\frac{a+b}{2} \geq \sqrt{ab}$. Hence prove that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ for positive a, b, c, d.

(b) Let $d = \frac{a+b+c}{3}$. Show that $abc \leq \left(\frac{a+b+c}{3}\right)^3$.

8 If $x > 0$ and $y > 0$, prove that:

(a) $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x + y}$

(b) $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{(x + y)^2}$

9 **(a)** If a and b are real numbers, prove that $4a^2 - 6ab + 4b^2 \geq a^2 + b^2$.

(b) Write the binomial expansion of $(a-b)^4$ and prove that $a^4 + b^4 \geq a^3b + ab^3$ if $a > 0$ and $b > 0$.

14 The area of a triangle is given by Heron's formula as $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b and c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$. Given that $\sqrt{ab} \leq \frac{a+b}{2}$ and $ab + bc + ca \leq a^2 + b^2 + c^2$, show that: $A \leq \frac{a^2+b^2+c^2}{6}$.

17 Let $g(x) = \sin x - x$.

(a) Show that $g(0) = 0$ and $g'(0) = 0$.

(b) Show that $-2 \le g'(x) \le 0$ for all x.

(c) Hence explain why $g(x) \le 0$ for $x \ge 0$.

(d) Explain why $\sin x < x$ for $x > 0$.

19 By letting $a = \frac{1}{x}$ and $b = \frac{1}{y}$ in $\frac{a+b}{2} \geq \sqrt{ab}$, prove that:

(a) $\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$

(b) $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{xy}$

20 If $1 \le x \le 4$, show that: $\frac{1}{3} \le \frac{1}{1+\sqrt{x}} \le \frac{1}{2}$