SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)\,g(y)$ USING SEPARATION OF VARIABLES

1 Use the method of separation of variables to find the general solution of each of the differential equations below. Where reasonable, express the family of solutions as explicit functions of x.

(a) $(x^2+4)\dfrac{dy}{dx} = 2xy$ (b) $\dfrac{dy}{dx} = \dfrac{2y}{x}$ (c) $\dfrac{dy}{dx} = (1+y^2)\sqrt{x}$

a) $\dfrac{dy}{y} = \dfrac{2x\,dx}{x^2+4} \Rightarrow \int \dfrac{dy}{y} = \int \dfrac{2x\,dx}{x^2+4}$ so $\ln|y| = \ln|x^2+4| + C$

So $\ln|y| = \ln|A(x^2+4)|$ so $y = A(x^2+4)$

b) $\dfrac{dy}{y} = 2\dfrac{dx}{x} \Rightarrow \int \dfrac{dy}{y} = 2\int \dfrac{dx}{x} \Rightarrow \ln|y| = 2\ln|x| + C$

$\Rightarrow \ln|y| = \ln x^2 + C$ so $\ln|y| = \ln x^2 + \ln A = \ln|Ax^2|$

$y = Ax^2$

c) $\dfrac{dy}{1+y^2} = \sqrt{x}\,dx = x^{1/2}\,dx$ so $\int \dfrac{dy}{1+y^2} = \int x^{1/2}\,dx$

$\tan^{-1} y = \dfrac{x^{1/2+1}}{1/2+1} + C = \dfrac{x^{3/2}}{3/2} + C = \dfrac{2}{3}x^{3/2} + C$

So $y = \tan\left[\dfrac{2}{3}x^{3/2} + C\right]$

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1 Use the method of separation of variables to find the general solution of each of the differential equations below. Where reasonable, express the family of solutions as explicit functions of x.

(e) $\left(1+x^2\right)\dfrac{dy}{dx} = xy$ **(f)** $e^y \cos x - \dfrac{dy}{dx}\sin^2 x = 0$ **(g)** $(\sec x)\,y' + y^2 = 0$

e) $\dfrac{dy}{y} = \dfrac{x\,dx}{1+x^2}$ so $\displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$

$\ln|y| = \frac{1}{2}\ln(1+x^2) + C = \ln\sqrt{1+x^2} + C = \ln\left|A\sqrt{1+x^2}\right|$

so $y = A\sqrt{1+x^2}$

f) $e^y \cos x = \dfrac{dy}{dx}\sin^2 x \Rightarrow \dfrac{dy}{e^y} = \dfrac{\cos x\,dx}{\sin^2 x}$

$\Rightarrow e^{-y}\,dy = \dfrac{\cos x}{\sin^2 x}\,dx \Rightarrow \displaystyle\int e^{-y}\,dy = \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{(\sin x)'}{\sin^2 x}\,dx$

$\Rightarrow -e^{-y} = \dfrac{-1}{\sin x} + C \Rightarrow e^{-y} = +\dfrac{1}{\sin x} - C = +\csc x - C$

so $-y = \ln|+\csc x - C| = \ln|\csc x + K|$

$y = -\ln|\csc x + K| = \ln\left|\dfrac{1}{\csc x + K}\right|$

g) $\sec x\,\dfrac{dy}{dx} = -y^2 \Rightarrow -\dfrac{dy}{y^2} = \dfrac{dx}{\sec x} = \cos x\,dx$

$-\displaystyle\int \frac{dy}{y^2} = \int \cos x\,dx \Rightarrow -\frac{y^{-2+1}}{(-2+1)} = \sin x + C$

So $y^{-1} = \sin x + C$ or $\dfrac{1}{y} = \sin x + C$

$y = \dfrac{1}{\sin x + C}$

2 Find the particular solution of $e^{-x^2} yy' + xy = 0$, $y(0) = 1$.

$e^{-x^2} yy' + xy = 0 \Rightarrow e^{-x^2} y' + x = 0$

or $e^{-x^2} \dfrac{dy}{dx} + x = 0 \Rightarrow e^{-x^2} dy = -x\,dx$

or $dy = -x e^{x^2} dx \Rightarrow \int dy = \int -x e^{x^2} dx$

$y = -\frac{1}{2} \int 2x e^{x^2} dx = -\frac{1}{2} e^{x^2} + C$

So the general solution is $y = -\frac{1}{2} e^{x^2} + C$

But the particular solution is for $y(0) = 1$

$y(0) = -\frac{1}{2} e^{0^2} + C = -\frac{1}{2} + C$ so $C = 1 + \frac{1}{2} = \frac{3}{2}$

The particular solution for which $y(0) = 1$ is

$$y = -\frac{1}{2} e^{x^2} + \frac{3}{2}$$

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3 Find the equation of each graph:

(a) The graph passes through (1, 2) and has a slope $\frac{3y}{x^2}$ at each point (x, y).

(b) The gradient of the tangent at point (x, y) on a graph is given by $\frac{-2y}{x}$ and the graph passes through the point (1, 2).

a) $\frac{dy}{dx} = \frac{3y}{x^2} \Rightarrow \frac{dy}{y} = 3\frac{dx}{x^2} \Rightarrow \int \frac{dy}{y} = 3\int \frac{dx}{x^2}$

so $\ln|y| = 3\frac{x^{-2+1}}{(-2+1)} + C = \frac{3x^{-1}}{(-1)} + C = -\frac{3}{x} + C$

So $y = Ae^{-3/x}$

For $x=1, y=2$ $\quad y(1) = Ae^{-3} = 2$ $\quad$ so $A = \frac{2}{e^{-3}} = 2e^3$

The particular solution is $y = 2e^3 e^{-3/x} = 2e^{(3-3/x)}$

b) $\frac{dy}{dx} = -\frac{2y}{x} \Rightarrow \frac{dy}{y} = -\frac{2\,dx}{x} \Rightarrow \int \frac{dy}{y} = -2\int \frac{dx}{x}$

$\ln|y| = -2\ln|x| + C = \ln|x^{-2}| + C = \ln\left(\frac{|A|}{x^2}\right)$

So $y = \frac{A}{x^2}$

$y(1) = \frac{A}{1^2} = A$ so as $y(1) = 2$ then $A = 2$

The particular solution for which (1,2) belongs to is

$y(x) = \frac{2}{x^2}$

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5 Consider the differential equation $\frac{dy}{dx} = 3x^2 \cos^2 y$.

(a) Find the particular solution $y = f(x)$ to the differential equation, satisfying the initial condition $f(0) = \frac{\pi}{4}$

(b) State the domain and range of the solution found in part **(a)**.

a) $\frac{dy}{dx} = 3x^2 \cos^2 y \Rightarrow \frac{dy}{\cos^2 y} = 3x^2\,dx$ or $\sec^2 y\,dy = 3x^2\,dx$

so $\int \sec^2 y\,dy = \int 3x^2\,dx \Rightarrow \tan y = x^3 + C$

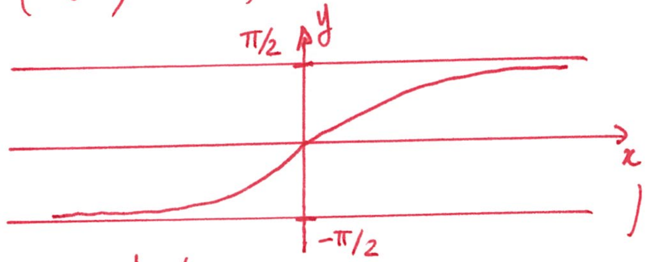
So $y = \tan^{-1}[x^3 + C]$

$f(0) = \tan^{-1}[0^3 + C] = \tan^{-1} C = \frac{\pi}{4}$ so $C = 1$

This particular solution is $f(x) = \tan^{-1}[x^3 + 1]$

b) $f(x) = \tan^{-1} x$ is defined on $(-\infty, +\infty)$

(as its graph is:



)

$(x^3 + 1)$ takes values between $-\infty$ and $+\infty$

when x takes values between $-\infty$ and $+\infty$

So the domain of $f(x) = \tan^{-1}(x^3 + 1)$ is $(-\infty, +\infty)$ (basically $\mathbb{R}$)

Range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

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6 An insect population P experiences a seasonal growth rate given by $\frac{dP}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right)P$, $P(0) = 1$, where P is measured in millions and t is the number of months since the beginning of spring.

Express the time variation of the insect population P and sketch this variation over the course of one year.

$$\frac{dP}{dt} = \frac{\pi}{12} \sin\left(\frac{\pi}{6}t\right) \times P \quad \text{so} \quad \frac{dP}{P} = \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right) dt$$

$$\Rightarrow \int \frac{dP}{P} = \int \frac{\pi}{12} \sin\left(\frac{\pi t}{6}\right) dt \Rightarrow \ln|P| = \frac{\pi}{12}\left(-\cos\left(\frac{\pi t}{6}\right) \times \frac{6}{\pi}\right) + C$$

$$\Rightarrow \ln|P| = -\frac{1}{2}\cos\left(\frac{\pi t}{6}\right) + C$$

$$\Rightarrow P = A\, e^{-\frac{1}{2}\cos\left(\frac{\pi t}{6}\right)}$$

$$P(0) = A\, e^{-\frac{1}{2}\cos 0} = A\, e^{-1/2} = 1 \quad \text{so} \quad A = \frac{1}{e^{-1/2}} = e^{1/2}$$

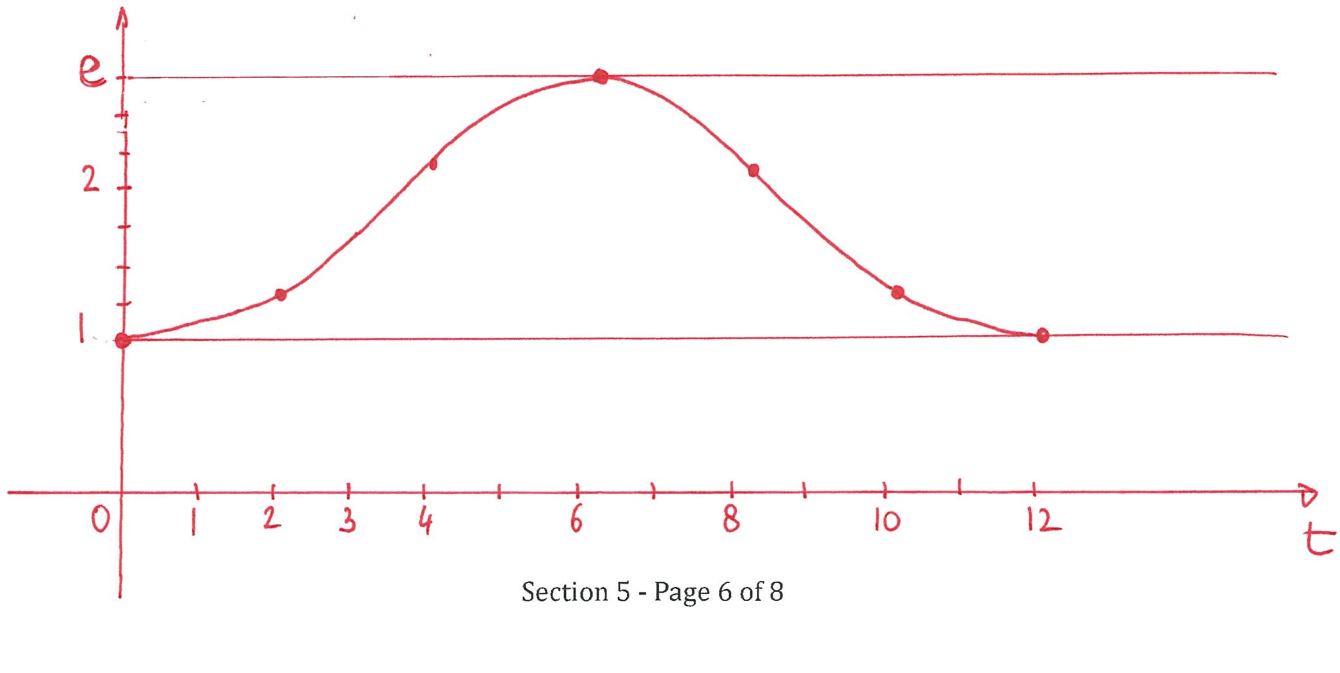
$\therefore$ this particular solution is $P(t) = e^{1/2} e^{-1/2 \cos\left(\frac{\pi t}{6}\right)}$

$$P(t) = e^{\frac{1}{2}\left(1 - \cos\left(\frac{\pi t}{6}\right)\right)}$$

But $\cos 2\theta = 1 - 2\sin^2\theta$

so $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

$$\therefore P(t) = e^{\sin^2\left(\frac{\pi t}{12}\right)}$$



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10 Consider the differential equation $\frac{dy}{dx} = -\frac{2xy}{\log_e y}$, $y > 0$.

(a) Find the general solution $g(x, y) = c$ of this differential equation as an implicit relation between x and y, using the substitution $u = \log_e y$ to complete the integration.

(b) Find the particular solution passing through the point $(0, e)$.

(c) Explain why $x = 1$ cannot exist in the solution to part (b).

a) $\frac{\ln y}{y}\,dy = -2x\,dx$ ①

Change of variable $u = \ln|y|$ so $\frac{du}{dy} = \frac{1}{y}$ or $du = \frac{dy}{y}$

$\therefore$ ① $\Leftrightarrow$ $u\,du = -2x\,dx$ $\Rightarrow \int u\,du = \int -2x\,dx$

$\Rightarrow \frac{u^2}{2} = -x^2 + C \Rightarrow u^2 = -2x^2 + D$

$\Rightarrow (\ln|y|)^2 = -2x^2 + D$

b) Point $(0, e)$ $\Rightarrow \underbrace{(\ln e)^2}_{=1} = -2 \times 0^2 + D$

so $D = 1$

This particular solution is $(\ln|y|)^2 = -2x^2 + 1$

$\Leftrightarrow (\ln|y|)^2 = 1 - 2x^2$

c) if $x = 1$ then $1 - 2 \times 1^2 = -1$ which is negative

that's impossible as the other side is always positive (being a square)

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14 A space probe is launched vertically upwards from the surface of a spherical planet with a radius R. If the atmospheric drag is ignored, the upwards velocity (v m s^{-1}) of the probe at height h metres above the surface of the planet is modelled by the solution of the differential equation $\frac{dv}{dh} = -\frac{gR^2}{v(R+h)^2}$, $v = u$, where $h = 0$ and g is the gravitational acceleration on the surface of the planet.

(a) Show that $v^2 = u^2 - \frac{2gR}{1+\frac{R}{h}}$.

(b) Hence find the minimum launch velocity u for the probe to escape the planet's gravity.

a) $\frac{dv}{dh} = -\frac{gR^2}{v(R+h)^2} \Rightarrow v\,dv = -gR^2 \times \frac{dh}{(R+h)^2}$

$\Rightarrow \int v\,dv = \int -gR^2 \frac{dh}{(R+h)^2} = -gR^2 \int \frac{dh}{(R+h)^2} = -gR^2 \int (R+h)^{-2}\,dh$

$\Rightarrow \frac{v^2}{2} = -gR^2 \frac{(R+h)^{-2+1}}{(-2+1)} + C = \frac{gR^2}{(R+h)} + C$

So $v^2 = \frac{2gR^2}{h+R} + C = \frac{2gR}{1+h/R} + C$

when $h = 0$, $v = u$

So $u^2 = 2gR + C$ so $C = u^2 - 2gR$

$\therefore v^2 = \frac{2gR}{1+h/R} + u^2 - 2gR = u^2 - 2gR\left[1 - \frac{1}{1+h/R}\right]$

$v^2 = u^2 - 2gR\left[\frac{1+h/R-1}{1+h/R}\right] = u^2 - 2gR\left[\frac{h/R}{1+h/R}\right]$

$\therefore v^2 = u^2 - 2gR\left[\frac{1}{R/h+1}\right] = u^2 - \frac{2gR}{1+R/h}$

b) To escape the planet's gravity, the upwards velocity v must be non-negative as $h \to +\infty$

$\lim_{R \to +\infty}\left(u^2 - \frac{2gR}{1+R/h}\right) = u^2 - 2gR$ which is positive when

$u^2 \geq 2gR$

or $u \geq \sqrt{2gR}$

Note: for Earth: $u \geq \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$ so $u \geq 11.2$ km s^{-1} That's fast!