SOME APPLICATIONS OF LOGARITHMS

A range of logarithmic scales are used in science and engineering to measure natural phenomena. When phenomena involve large differences in values, logarithms of values can be used to reduce the numbers to a size that is meaningful to non-scientific people so that useful comparisons can be made. Sometimes the common logarithm base 10 is used; at other times base 2 is more useful.

Logarithmic scales, rather than linear scales, are also used to graph experimental results. If this produces a straight line graph, then this indicates a logarithmic mathematical relationship.

Before calculators were invented, tables of logarithms and devices called 'slide rules' (see the picture at right) were used to perform calculations involving multiplication, division and corresponding operations, e.g. raising to a power or finding the square root. The slide rule had a logarithmic scale and used the fact that you can find the product of two numbers by adding their indices or logarithms.

Decibels (dB)

The decibel was originally used to measure sound levels, but now is also widely used as a measurement unit in electronics, signals and communications. It is based on a logarithmic scale. The difference in intensity or 'loudness' between two sounds, or between two sources of power P_1 and P_2, is defined to be $10\log_{10}\left(\frac{P_2}{P_1}\right)$ dB. The absolute measurement of the intensity or loudness of a sound is given by $L = 10\log_{10}\left(\frac{P}{P_0}\right)$, where P_0 is the reference value and has an intensity of 0 dB.

This is the quietest noise that can be (just) heard by the typical human ear. It has an intensity of 10^{-12} W/m^2 (watts per square metre). The intensity of any other sound measured in these units is

$$\begin{aligned}10\log_{10}\left(\frac{P}{P_0}\right) &= 10\log_{10} P - 10\log_{10} P_0\\ &= 10\log_{10} P - 0\\ &= 10\log_{10} P\end{aligned}$$

Sounds quieter than P_0 have a negative number of decibels.

Because it is a logarithmic scale, every increase of 10 dB is 10 times more powerful. So if 0 dB the smallest audible sound, then 10 dB is a sound 10 times more powerful than that, 20 dB is a sound 100 times more powerful, and 30 dB is 1000 times more powerful.

Sound intensities for some common events, in dB:

- near total silence: 0 dB
- a normal conversation: 60 dB
- a lawnmower: 90 dB
- a rock concert or a jet engine: 120 dB
- a gunshot or a firecracker: 140 dB and above.

You should not be exposed to a continuous noise level of 85 dB for more than 8 hours, or else you risk hearing damage. For each 3 dB increase, the length of time that you can be exposed is halved. This means that if the noise level in a nightclub is 100 dB, then you should only stay 15 minutes. Of course, the further away from the noise source you are, the lower the sound intensity.

The decibel scale reduces very large numbers to numbers that people can relate to.

Example 21

(a) **(i)** How many times more intense is a sound P_2 which is 30 dB louder than another sound P_1?

(ii) How many times more intense is a sound P_2 which is 14 dB louder than another sound P_3?

(b) How much louder is:

(i) a sound of 50 dB than the smallest audible sound

(ii) a sound of 47 dB than a sound of 27 dB

(iii) a sound of 102 dB than a sound of 64 dB?

Solution

(a) **(i)**

$$10\log_{10}\left(\frac{P_2}{P_1}\right) = 30$$

$$\log_{10}\left(\frac{P_2}{P_1}\right) = 3$$

$$\frac{P_2}{P_1} = 10^3 = 1000$$

P_2 is 1000 times as loud as P_1.

(ii)

$$10\log_{10}\left(\frac{P_2}{P_3}\right) = 14$$

$$\log_{10}\left(\frac{P_2}{P_3}\right) = 1.4$$

$$\frac{P_2}{P_3} = 10^{1.4} \approx 25$$

P_2 is about 25 times as loud as P_3.

(b) **(i)** The smallest audible sound is 0 dB

50 dB − 0 dB = 50 dB

$$10\log_{10}\left(\frac{P}{P_0}\right) = 50$$

$$\log_{10}\left(\frac{P}{P_0}\right) = 5$$

$$\frac{P}{P_0} = 10^5$$

$$P = 10^5 P_0$$

The sound is 100 000 times as loud.

(ii) 47 dB − 27 dB = 20 dB

$$10\log_{10}\left(\frac{P}{P_0}\right) = 20$$

$$\log_{10}\left(\frac{P}{P_0}\right) = 2t$$

$$\frac{P}{P_0} = 10^2$$

$$P = 100\,P_0$$

The sound is 100 times as loud.

(iii) 102 dB − 64 dB = 38 dB

$$10\log_{10}\left(\frac{P}{P_0}\right) = 38$$

$$\log_{10}\left(\frac{P}{P_0}\right) = 3.8$$

$$\frac{P}{P_0} = 10^{3.8}$$

$$P \approx 6300\,P_0$$

The sound is 6300 times as loud.

SOME APPLICATIONS OF LOGARITHMS

Richter scale

The Richter scale is a logarithmic scale that is used to compare the magnitude of earthquakes. It was developed in the 1930s in California by Charles Richter. Using the Richter scale, the magnitude of an earthquake is expressed as a number (without units) determined by the logarithm of the amplitude of waves recorded by seismographs, which are devices that directly measure the seismic vibrations of the earth. Adjustments are made for the variations in distance between the various seismographs and the epicentre of the earthquake.

The Richter scale formula is $M_L = \log_{10}\left(\frac{A}{A_0}\right)$, where M_L is the measurement value of the Richter scale, A is the amplitude of the wave recorded by the seismograph and A_0 is the reference value that corresponds to a zero-level, earthquake, i.e. no earthquake. Because the scale is logarithmic, every increase of 1 on the scale corresponds to an earthquake that is 10 times 'stronger'. This means that an earthquake that measures 6.3 on the Richter scale is 10 times as strong as an earthquake measuring 5.3. This corresponds to a tenfold increase in the wave amplitude, but this actually means it releases approximately 32 times more energy.

The difference on the Richter scale between the intensity (amplitude) of two earthquakes is $\log_{10}\left(\frac{A_2}{A_1}\right)$.

pH

The pH level of a solution (where the word 'solution' here means water that has something dissolved in it) is a measure of how acidic or basic the solution is. The more acidic a solution, the more positive hydrogen ions are produced for chemical reactions; the more basic a solution, the more negative hydroxide ions are produced for chemical reactions. Acids and bases are important because they are involved in many common chemical reactions that involve water and other useful compounds. pH levels vary from 0 to 14, with 7 being neutral (neither acidic nor basic); a pH of less than 7 indicates acidity, whereas a pH greater than 7 indicates basicity.

The definition of pH is given by $\text{pH} = -\log_{10}[H^+]$, where $[H^+]$ is the concentration of the H^+ (hydrogen) ions measured in units of mol/L.

In pure water, $[H^+] = 1.0 \times 10^{-7}$ mol/L, so the pH is $-\log_{10}[H^+] = -\log_{10}(1.0 \times 10^{-7}) = 7$.

Octaves

In music, there are 8 notes in each octave, named using the letters from A to G. An octave is any sequence of 8 notes starting and ending at the same note, e.g. C D E F G A B C.

Every musical note is a sound wave that has a particular vibration frequency. 'Higher'-sounding notes have higher frequency values, while 'lower' (or 'deeper') notes have lower frequency values.

The note at the high end of any octave has double the frequency of the note at the lower end. For every octave higher, the frequency doubles, so for two octaves higher the frequency would quadruple; three octaves higher, the frequency would increase by a factor of 8; and so on. This suggests using logarithms to the base 2 as a measure of how much higher one note is than another.

So, $\log_2\left(\frac{f_2}{f_1}\right)$ is a measure of how many octaves higher a note of frequency f_2 is than a note of frequency f_1.

For example, if two notes have frequencies of 120 Hz and 1920 Hz, then the difference can be measured as

$\log_2\left(\frac{1920}{120}\right) = \log_2 16 = 4$, meaning that one note is 4 octaves higher than the other.