FURTHER INTEGRATION - CHAPTER REVIEW

1 Evaluate: **(a)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\cos x \, dx$ **(b)** $\int_3^4 \frac{5x-7}{x^2-3x+2} dx$

a) we integrate by parts

$u(x) = x \qquad u'(x) = 1$

$v(x) = +\sin x \qquad v'(x) = \cos x$

$$\text{So } I = \Big[x\sin x\Big]_{\pi/6}^{\pi/3} - \int_{\pi/6}^{\pi/3} \sin x \, dx = \left[\frac{\pi}{3}\times\frac{\sqrt{3}}{2} - \frac{\pi}{6}\times\frac{1}{2}\right] - \Big[-\cos x\Big]_{\pi/6}^{\pi/3}$$

$$I = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} + \left[\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right] = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{\pi}{6}\left[\sqrt{3} - \frac{1}{2}\right] + \frac{1-\sqrt{3}}{2}$$

b) $$\frac{5x-7}{x^2-3x+2} = \frac{a}{x-1} + \frac{b}{x-2} = \frac{x(a+b) - 2a + (-b)}{(x-1)(x-2)}$$

$$\text{So } \begin{cases} a+b = 5 & ① \\ -2a - b = -7 & ② \end{cases} \Leftrightarrow \begin{cases} -a = -2 \\ a+b = 5 \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ b = 3 \end{cases}$$

$$\int_3^4 \frac{5x-7}{x^2-3x+2} dx = \int_3^4 \frac{2}{x-1} + \frac{3}{x-2} dx$$

$$= 2\Big[\ln(x-1)\Big]_3^4 + 3\Big[\ln(x-2)\Big]_3^4$$

$$= 2\left(\ln 3 - \ln 2\right) + 3\left(\ln 2 - \ln 1\right)$$

$$= 2\ln 3 - 2\ln 2 + 3\ln 2$$

$$= 2\ln 3 + \ln 2$$

$$= \ln 3^2 + \ln 2 = \ln 9 + \ln 2 = \ln 18$$

2 Find: (a) $\int \log_e 2x\,dx$ (b) $\int \frac{x+2}{x^2-1}\,dx$

a) We integrate by parts.

$u(x) = \ln 2x \qquad u'(x) = \frac{1}{2x} \times 2 = \frac{1}{x}$

$v(x) = x \qquad v'(x) = 1$

$$\int \ln 2x\,dx = x \ln 2x - \int x \times \frac{1}{x}\,dx = x \ln 2x - x + C$$

b) $$\int \frac{x+2}{x^2-1}\,dx = \int \frac{x+2}{(x-1)(x+1)}\,dx.$$

$$\frac{x+2}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{b}{x+1} = \frac{x(a+b)+a-b}{x^2-1}$$

So $\begin{cases} a+b=1 \\ a-b=2 \end{cases} \iff \begin{cases} 2a=3 \\ a+b=1 \end{cases} \iff \begin{cases} a=3/2 \\ b=1-3/2=-1/2 \end{cases}$

$$\int \frac{x+2}{(x-1)(x+1)}\,dx = \int \frac{3/2}{x-1} + \frac{(-1/2)}{x+1}\,dx$$

$$\_\_\_\_\_\_ = \frac{3}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1}$$

$$\_\_\_\_\_\_ = \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

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4 Evaluate: **(a)** $\int_1^{\sqrt{3}} \tan^{-1} x\, dx$ **(b)** $\int_{-2}^{2} \frac{6}{9-x^2} dx$

a) we integrate by parts

$u(x) = \tan^{-1}x \qquad u'(x) = \frac{1}{1+x^2}$

$v(x) = x \qquad v'(x) = 1$

$I = \left[x \tan^{-1}x\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{x}{1+x^2} dx$

$I = \left[\sqrt{3}\tan^{-1}\sqrt{3} - \tan^{-1}1\right] - \frac{1}{2}\int_1^{\sqrt{3}} \frac{2x}{x^2+1} dx$

$I = \sqrt{3}\frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$

$I = \pi\left(\frac{\sqrt{3}}{3} - \frac{1}{4}\right) - \frac{1}{2}\left(\ln(3+1) - \ln 2\right) = \pi\left(\frac{\sqrt{3}}{3} - \frac{1}{4}\right) - \ln\sqrt{2}$

b) This is an even function, $\therefore$ $I = 2\int_0^2 \frac{6}{9-x^2} dx$

$I = 12\int_0^2 \frac{1}{(3-x)(3+x)} dx = 12\left[\int_0^2 \frac{a}{3-x} + \frac{b}{3+x} dx\right]$

with $\begin{cases} a-b=0 \\ 3a+3b=1 \end{cases} \iff \begin{cases} a=b \\ a=1/6=b \end{cases}$

$I = 12 \times \frac{1}{6}\left[\int_0^2 \frac{1}{3-x} dx + \int_0^2 \frac{1}{x+3} dx\right]$

$I = 2\left[\left[-\ln(3-x)\right]_0^2 + \left[\ln[x+3)\right]_0^2\right]$

$I = 2\left[-\ln 1 + \ln 3 + \ln 5 - \ln 3\right]$

$I = 2\ln 5$

5 Find the derivative of $\log_e(\operatorname{cosec} x + \cot x)$ and deduce the value of:

(a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}\frac{\theta}{2}\, d\theta$ **(b)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec u\, du$

$$f(x) = \ln\left[\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right] = \ln\left[\frac{1+\cos x}{\sin x}\right]$$

$$\text{so } f'(x) = \frac{1}{\frac{1+\cos x}{\sin x}} \times \left(\frac{-\sin x \times \sin x - \cos x(1+\cos x)}{\sin^2 x}\right)$$

$$f'(x) = \frac{\sin x}{1+\cos x} \times \left[\frac{-\sin^2 x - \cos^2 x - \cos x}{\sin^2 x}\right]$$

$$f'(x) = \frac{1}{1+\cos x} \times \left[\frac{-1-\cos x}{\sin x}\right] = -\frac{1}{\sin x} = -\operatorname{cosec} x.$$

a) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}\frac{\theta}{2}\, d\theta = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x \times 2\,dx$ $\quad x = \frac{\theta}{2}$ so $\frac{dx}{d\theta} = \frac{1}{2}$, $d\theta = 2\,dx$

$$= -2\int_{\pi/6}^{\pi/4} (-\operatorname{cosec} x)\, dx = -2\left[\ln(\operatorname{cosec} x + \operatorname{cotan} x)\right]_{\pi/6}^{\pi/4}$$

$$= -2 \ln\left[\frac{\operatorname{cosec} \pi/4 + \operatorname{cotan} \pi/4}{\operatorname{cosec} \pi/6 + \operatorname{cotan} \pi/6}\right] = 2\ln\left[\frac{2+\sqrt{3}}{\sqrt{2}+1}\right]$$

b) $\int_{\pi/6}^{\pi/3} \sec u\, du = \left[\ln(\sec u + \tan u)\right]_{\pi/6}^{\pi/3}$ following the same model

$$= \ln\left[\frac{\sec \pi/3 + \tan \pi/3}{\sec \pi/6 + \tan \pi/6}\right]$$

$$= \ln\left[\frac{2+\sqrt{3}}{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}\right] = \ln\left[\frac{2+\sqrt{3}}{\frac{3}{\sqrt{3}}}\right] = \ln\left[\frac{2+\sqrt{3}}{\sqrt{3}}\right]$$

6 **(a)** Find the derivative of: $\dfrac{\sin x}{1-\sin^2 x}+\log_e\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ $= f(x) = \dfrac{\sin x}{\cos^2 x} + \dfrac{1}{2}\left[\ln(1+\sin x) - \ln(1-\sin x)\right]$

(b) Hence evaluate: $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^3\theta\, d\theta$

a) $$f'(x) = \frac{\cos x \times \cos^2 x - 2\cos x(-\sin x)\sin x}{\cos^4 x} + \frac{1}{2}\left[\frac{\cos x}{1+\sin x} - \frac{-\cos x}{1-\sin x}\right]$$

$$f'(x) = \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} + \frac{1}{2}\left[\frac{\cos x(1-\sin x) + \cos x(1+\sin x)}{(1+\sin x)(1-\sin x)}\right]$$

$$f'(x) = \frac{1+\sin^2 x}{\cos^3 x} + \frac{1}{2}\,\frac{2\cos x}{1-\sin^2 x}$$

$$f'(x) = \frac{1+\sin^2 x}{\cos^3 x} + \frac{\cos x}{\cos^2 x} = \frac{1+\sin^2 x}{\cos^3 x} + \frac{\cos^2 x}{\cos^3 x}$$

$$f'(x) = \frac{1+\sin^2 x + \cos^2 x}{\cos^3 x} = \frac{2}{\cos^3 x} = 2\sec^3 x.$$

b) $$\int_{\pi/6}^{\pi/3} \sec^3\theta\, d\theta = \frac{1}{2}\left[\frac{\sin x}{1-\sin^2 x} + \ln\sqrt{\frac{1+\sin x}{1-\sin x}}\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin \pi/3}{1-\sin^2 \pi/3} + \ln\sqrt{\frac{1+\sin\pi/3}{1-\sin\pi/3}}\right] - \left[\frac{\sin\pi/6}{1-\sin^2\pi/6} + \ln\sqrt{\frac{1+\sin\pi/6}{1-\sin\pi/6}}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\sqrt{3}/2}{1-3/4} + \ln\sqrt{\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}}\right] - \left[\frac{1/2}{1-1/4} + \ln\sqrt{\frac{1+1/2}{1-1/2}}\right]\right\}$$

$$= \frac{1}{2}\left\{2\sqrt{3} + \ln\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} - \frac{2}{3} - \ln\sqrt{3}\right\}$$

$$= \frac{1}{2}\left\{2\sqrt{3} - \frac{2}{3} + \ln\sqrt{\frac{(2+\sqrt{3})^2}{1}} - \ln\sqrt{3}\right\}$$

$$= \frac{1}{2}\left[2\sqrt{3} - \frac{2}{3} + \ln\sqrt{\frac{7+2\sqrt{3}}{3}}\right]$$

7 **(a)** Write $(x-1)(7-x)$ in the form $b^2-(x-a)^2$, where a and b are real numbers.

(b) Using the values of a and b from part **(a)** and making the substitution $x-a=b\sin\theta$, or otherwise, evaluate: $\int_1^7 \sqrt{(x-1)(7-x)}\,dx$

a) $b^2-(x-a)^2 = b^2-(x^2-2ax+a^2) = -x^2+2ax+b^2-a^2$

$(x-1)(7-x) = -x^2+7x+x-7 = -x^2+8x-7$

So $2a=8$ i.e. $a=4$

and $b^2-a^2=-7$ so $b^2-16=-7$ so $b^2=9$ $b=\pm 3$

b) $\int_1^7 \sqrt{(x-1)(7-x)}\,dx = \int_1^7 \sqrt{3^2-(x-4)^2}\,dx$

let $x = 3\sin\theta + 4$ so $\frac{dx}{d\theta} = 3\cos\theta$ or $dx = 3\cos\theta\,d\theta$

$x-4 = 3\sin\theta$

when $x=1$, $-3 = 3\sin\theta$ so $\sin\theta=-1$ $\theta = -\pi/2$

when $x=7$ $7-4=3=3\sin\theta$ so $\sin\theta = 1$ $\theta=\pi/2$

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{3^2-(3\sin\theta)^2} \times 3\cos\theta\,d\theta$$

$$I = \int_{-\pi/2}^{\pi/2} 3\sqrt{1-\sin^2\theta} \times 3\cos\theta\,d\theta = 9\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$$I = 9\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}\,d\theta = \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\pi/2}^{\pi/2}$$

$$I = \frac{9}{2}\left[\left(0+\frac{\pi}{2}\right) - \left(0 - \frac{\pi}{2}\right)\right] = \frac{9}{2}\times\pi = \frac{9\pi}{2}$$

9 Reduce each rational function to its partial fractions.

(a) $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ **(b)** $\dfrac{x^2+10x+16}{(x-1)(x^2-4)}$

a) For x^2-5x+6 , $\Delta = 25-4\times 6 = 1$ $r_1 = \dfrac{5+1}{2} = 3$ $r_2 = \dfrac{5-1}{2} = 2$

$$\text{So } \frac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{x-3}$$

$$\text{——} = \frac{a(x-2)(x-3)+b(x-1)(x-3)+c(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\begin{cases} a+b+c=1 \\ -5a-4b-3c=-10 \\ 6a+3b+2c=13 \end{cases} \Leftrightarrow \begin{cases} 2a+2b+2c=2 \\ 6a+3b+2c=13 \\ +5a+4b+3c=+10 \end{cases} \Leftrightarrow \begin{cases} 4a+b=11 \\ a+b+c=1 \\ 5a+4b+3c=10 \end{cases}$$

$$\Leftrightarrow \begin{cases} b=11-4a \\ 11-3a+c=1 \Leftrightarrow c=3a-10 \\ 5a+4(11-4a)+3(3a-10)=10 \end{cases} \Leftrightarrow \begin{cases} -2a=-4 \quad \text{so } a=2 \\ b=3 \\ c=-4 \end{cases}$$

$$\text{b) } \frac{x^2+10x+16}{(x-1)(x^2-4)} = \frac{x+8}{(x-1)(x-2)} = \frac{a}{x-1} + \frac{b}{x-2}$$

$$\text{so } \text{——} = \frac{a(x-2)+b(x-1)}{(x-1)(x-2)}$$

$$\begin{cases} a+b=1 \\ -2a-b=8 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ -2a-(1-a)=8 \end{cases} \Leftrightarrow \begin{cases} b=1-a \\ -a=8+1=9 \end{cases} \quad \begin{cases} a=-9 \\ b=10 \end{cases}$$

$$\text{So } \frac{x^2+10x+16}{(x-1)(x^2-4)} = \frac{-9}{x-1} + \frac{10}{x-2}$$

13 Evaluate:

(a) $\int_1^3 \frac{2x^2+2x+5}{(x^2+3)(2x-1)}dx$ (b) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\cos x\,dx$

a) $\frac{2x^2+2x+5}{(x^2+3)(2x-1)} = \frac{ax+b}{x^2+3} + \frac{c}{2x-1}$

$\Rightarrow \begin{cases} 2a+c=2 \\ -a+2b=2 \\ -b+3c=5 \end{cases} \Rightarrow \begin{cases} 2a+c=2 \\ -2a+4b=4 \\ -b+3c=5 \end{cases} \Rightarrow \begin{cases} 4b+c=6 \\ -4b+12c=20 \\ 2a+c=2 \end{cases}$

$\Rightarrow \begin{cases} 13c=26 \text{ so } c=2 \\ 4b=12\times 2-20=4 \text{ so } b=1 \\ a=2b-2=2\times 1-2=0 \end{cases}$ $a=0,\ b=1,\ c=2$

$\int_1^3 \frac{2x^2+2x+5}{(x^2+3)(2x-1)}dx = \int_1^3 \frac{dx}{x^2+3} + \int_1^3 \frac{2}{2x-1}dx$

$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3 + \left[\ln\left(x-\frac{1}{2}\right)\right]_1^3$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] + \ln\left(\frac{5/2}{1/2}\right) = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] + \ln 5$

$= \frac{\pi}{6\sqrt{3}} + \ln 5$

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b) we integrate by parts

$u(x) = x$ $u'(x) = 1$

$v(x) = \sin x$ $v'(x) = \cos x$

$I = \left[x\sin x\right]_{\pi/4}^{3\pi/4} - \int_{\pi/4}^{3\pi/4} \sin x\,dx$

$I = \left[\frac{3\pi}{4}\sin\frac{3\pi}{4} - \frac{\pi}{4}\sin\frac{\pi}{4}\right] - \left[-\cos x\right]_{\pi/4}^{3\pi/4}$

$I = \frac{3\pi}{4}\times\frac{\sqrt{2}}{2} - \frac{\pi}{4}\times\frac{\sqrt{2}}{2} + \left[\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}\right] = \frac{3\pi\sqrt{2}}{8} - \frac{\pi\sqrt{2}}{8} + \left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}$

$I = \frac{\pi\sqrt{2}}{4} - \sqrt{2} = \sqrt{2}\left[\frac{\pi}{4} - 1\right]$

14 Differentiate $\log_e x - \log_e\left(a+\sqrt{a^2-x^2}\right)$ where $a>0$ and deduce the value of: $\int_3^4 \frac{dx}{x\sqrt{25-x^2}}$

$$f(x) = \ln x - \ln\left[a+\sqrt{a^2-x^2}\right]$$

$$f'(x) = \frac{1}{x} - \frac{1}{a+\sqrt{a^2-x^2}} \times \left[\frac{1}{2}(a^2-x^2)^{-1/2} \times (-2x)\right]$$

$$f'(x) = \frac{1}{x} + \frac{1}{a+\sqrt{a^2-x^2}} \times \frac{x}{\sqrt{a^2-x^2}}$$

$$f'(x) = \frac{1}{x} + \frac{x}{a\sqrt{a^2-x^2}+a^2-x^2}$$

$$f'(x) = \frac{a\sqrt{a^2-x^2}+a^2-x^2+x^2}{x\left[a\sqrt{a^2-x^2}+a^2-x^2\right]} = \frac{a\left[\sqrt{a^2-x^2}+a\right]}{x\left[a\sqrt{a^2-x^2}+a^2-x^2\right]}$$

$$f'(x) = \frac{a\left[\cancel{\sqrt{a^2-x^2}+a}\right]}{x\sqrt{a^2-x^2}\left[\cancel{a+\sqrt{a^2-x^2}}\right]} = \frac{a}{x\sqrt{a^2-x^2}}$$

$$\text{So } \int_3^4 \frac{dx}{x\sqrt{25-x^2}} = \frac{1}{5}\int_3^4 \frac{5\,dx}{x\sqrt{5^2-x^2}}$$

$$\int_3^4 \frac{dx}{x\sqrt{25-x^2}} = \frac{1}{5}\left[\ln x - \ln\left(5+\sqrt{5^2-x^2}\right)\right]_3^4$$

$$= \frac{1}{5}\left[\left[\ln 4 - \ln(5+3)\right] - \left[\ln 3 - \ln(5+4)\right]\right]$$

$$= \frac{1}{5}\left[\ln 4 - \ln 8 - \ln 3 + \ln 9\right]$$

$$= \frac{1}{5}\ln\left(\frac{4\times 9}{8\times 3}\right) = \frac{1}{5}\ln\left(\frac{3}{2}\right)$$

15 Evaluate: **(a)** $\int_1^4 (x+1)\sqrt{x}\,dx$ **(b)** $\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\,dx$ **(c)** $\int_5^6 \frac{dx}{x^2-16}$

a) $\int_1^4 x\sqrt{x} + \sqrt{x}\,dx = \int_1^4 x^{3/2} + x^{1/2}\,dx = \left[\frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2}\right]_1^4$

$= \left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right]_1^4 = 2\left[\frac{4^{5/2}}{5} + \frac{4^{3/2}}{3} - \frac{1}{5} - \frac{1}{3}\right]$

$= 2\left[\frac{2^5}{5} + \frac{2^3}{3} - \frac{8}{15}\right] = 2\left[\frac{32}{5} + \frac{8}{3} - \frac{8}{15}\right] = \frac{256}{15} = 17\frac{1}{15}$

b) $\int_0^{\pi/2} \cos x\, e^{\sin x}\,dx = \int_0^{\pi/2} (\sin x)'\, e^{\sin x}\,dx = \left[e^{\sin x}\right]_0^{\pi/2}$

$= e^{\sin \pi/2} - e^{\sin 0} = e^1 - e^0 = e - 1$

c) $\frac{1}{x^2-16} = \frac{a}{x-4} + \frac{b}{x+4}$

$\begin{cases} a+b=0 \\ 4a-4b=1 \end{cases} \Leftrightarrow \begin{cases} 4a+4b=0 \\ 4a-4b=1 \end{cases} \Leftrightarrow \begin{cases} a=1/8 \\ b=-1/8 \end{cases}$, so

$\int_5^6 \frac{dx}{x^2-16} = \int_5^6 \frac{1/8}{x-4} - \frac{1/8}{x+4}\,dx$

$= \frac{1}{8}\left[\int_5^6 \frac{dx}{x-4} - \int_5^6 \frac{dx}{x+4}\right]$

$= \frac{1}{8}\left[\left[\ln(x-4)\right]_5^6 - \left[\ln(x+4)\right]_5^6\right]$

$= \frac{1}{8}\left[(\ln 2 - \ln 1) - (\ln 10 - \ln 9)\right]$

$= \frac{1}{8}\left[\ln 2 - \ln 10 + \ln 9\right] = \frac{1}{8}\ln\frac{18}{10} = \frac{1}{8}\ln\left(\frac{9}{5}\right)$

15 Evaluate: **(d)** $\int_{-1}^{1}(2x-1)\sin x\,dx$ **(e)** $\int_{0}^{3}x^2\sqrt{9-x^2}\,dx$

d) $\int_{-1}^{1}(2x\sin x)\,dx - \underbrace{\int_{-1}^{1}\sin x\,dx}_{=0 \text{ as } \sin x \text{ is odd.}} = 2\int_{-1}^{1} x\sin x\,dx$ ← which we integrate by parts.

$u(x) = x \qquad u'(x) = 1$

$v(x) = -\cos x \qquad v'(x) = \sin x$

$I = 2\left[\left[-x\cos x\right]_{-1}^{1} - \int_{-1}^{1}(-\cos x)\,dx\right] = 2\left[\left[x\cos x\right]_{1}^{-1} + \int_{-1}^{1}\cos x\,dx\right]$

$I = 2\left[-\cos(-1) - \cos 1 + \left[\sin x\right]_{-1}^{1}\right] = 2\left[-2\cos 1 + \sin 1 - \sin(-1)\right]$

$I = 2\left[-2\cos 1 + \sin 1 + \sin 1\right] = 4\left[\sin 1 - \cos 1\right]$

e) $\int_{0}^{3} x^2\sqrt{9-x^2}\,dx \qquad x = 3\sin\theta \quad \text{so} \quad \frac{dx}{d\theta} = 3\cos\theta \quad dx = 3\cos\theta\,d\theta$

$I = \int_{0}^{\pi/2} 9\sin^2\theta\,\sqrt{9-9\sin^2\theta} \times 3\cos\theta\,d\theta$

$I = 81\int_{0}^{\pi/2}\sin^2\theta\,\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = 81\int_{0}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$

$I = 81\int_{0}^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{81}{4}\int_{0}^{\pi/2}\sin^2 2\theta\,d\theta \qquad \cos 2x = 1 - 2\sin^2 x$

$I = \frac{81}{4}\int_{0}^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta = \frac{81}{8}\int_{0}^{\pi/2}(1-\cos 4\theta)\,d\theta$

$I = \frac{81}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_{0}^{\pi/2} = \frac{81}{8}\left(\frac{\pi}{2}\right) = \frac{81\pi}{16}$

16 Find: **(a)** $\int \frac{dx}{1-4x^2}$ **(b)** $\int \frac{x}{1-4x^2}dx$ **(c)** $\int \frac{x^2}{1-4x^2}dx$

a) $\frac{1}{1-4x^2} = \frac{a}{1-2x} + \frac{b}{1+2x}$

$\begin{cases} 2a - 2b = 0 \\ a + b = 1 \end{cases}$ $\begin{cases} a = b \\ 2a = 1 \end{cases}$ so $\begin{cases} a = 1/2 \\ b = 1/2 \end{cases}$

$\int \frac{dx}{1-4x^2} = \int \frac{1/2 \, dx}{1-2x} + \int \frac{1/2 \, dx}{1+2x} = \frac{1}{2}\left[\frac{\ln|1-2x|}{(-2)} + \frac{\ln|1+2x|}{2}\right] + C$

—— $= \frac{1}{4}\left[\ln|1+2x| - \ln|1-2x|\right] + C = \frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right| + C$

b) $\int \frac{x}{1-4x^2}dx = -\frac{1}{8}\int \frac{-8x}{1-4x^2}dx = -\frac{1}{8}\ln|1-4x^2| + C$

c) $\int \frac{x^2}{1-4x^2}dx = -\frac{1}{4}\int \frac{-4x^2}{1-4x^2}dx$

—— $= -\frac{1}{4}\left[\int \frac{1-4x^2-1}{1-4x^2}dx\right] = -\frac{1}{4}\left[\int 1 - \frac{1}{1-4x^2}dx\right]$

—— $= -\frac{x}{4} + \frac{1}{4}\int \frac{1}{1-4x^2}dx$ which we find at a)

—— $= -\frac{x}{4} + \frac{1}{4} \times \frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right| + C$

—— $= -\frac{x}{4} + \frac{1}{16}\ln\left|\frac{1+2x}{1-2x}\right| + C$

16 Find: (d) $\int \frac{x}{\sqrt{1-4x^2}}dx$ (e) $\int \frac{dx}{\sqrt{1-4x^2}}$ (f) $\int \frac{dx}{1+4x^2}$

d) $\int \frac{x}{\sqrt{1-4x^2}}\,dx = \frac{1}{8}\int \frac{8x}{\sqrt{1-4x^2}}\,dx = \frac{1}{8}\int 8x\left[1-4x^2\right]^{-1/2} dx$

$\text{———} = \frac{1}{8}\frac{(1-4x^2)^{1/2}}{-1/2} + C = -\frac{1}{4}(1-4x^2)^{1/2} + C = -\frac{\sqrt{1-4x^2}}{4} + C$

e) $\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1^2-(2x)^2}}$

$\text{———} = \frac{1}{2}\sin^{-1}\frac{2x}{1} + C = \frac{1}{2}\sin^{-1}2x + C$

f) $\int \frac{dx}{1+4x^2} = \int \frac{dx}{1^2+(2x)^2} = \frac{1}{2}\int \frac{2\,dx}{1^2+(2x)^2}$

$\text{———} = \frac{1}{2} \times \frac{1}{1}\tan^{-1}\left(\frac{2x}{1}\right) + C$

$\text{———} = \frac{1}{2}\tan^{-1}(2x) + C$

17 Find: (a) $\int \frac{dx}{\sin x + \tan x}$ (b) $\int \frac{dx}{5 + 4\cos 2x}$

$x = 2\tan^{-1} t$

a) if $t = \tan \frac{x}{2}$ $\sin x = \frac{2t}{1+t^2}$ $\tan x = \frac{2t}{1-t^2}$ $\frac{dx}{dt} = \frac{2}{1+t^2}$

$$I = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} = \int \frac{1-t^2}{t(1-t^2) + t(1+t^2)}\,dt = \int \frac{1-t^2}{2t}\,dt$$

$$I = \frac{1}{2}\left[\ln t - \frac{t^2}{2}\right] + C = \frac{1}{2}\left[\ln\left|\tan\frac{x}{2}\right| - \frac{1}{2}\tan^2\left(\frac{x}{2}\right)\right] + C$$

So $$\int \frac{dx}{\sin x + \tan x} = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\left(\frac{x}{2}\right) + C$$

b) if $t = \tan x$ then $\cos 2x = \frac{1-t^2}{1+t^2}$

$x = \tan^{-1} t$ $\frac{dx}{dt} = \frac{1}{1+t^2}$ so $dx = \frac{dt}{1+t^2}$

$$\int \frac{dx}{5 + 4\cos 2x} = \int \frac{\frac{1}{1+t^2}}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)}\,dt = \int \frac{dt}{5(1+t^2) + 4(1-t^2)}$$

$$— = \int \frac{dt}{t^2 + 9} = \int \frac{dt}{3^2 + t^2} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$— = \frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right) + C$$

So $$\int \frac{dx}{5 + 4\cos 2x} = \frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right) + C$$

17 Find: **(c)** $\int \frac{d\theta}{4\cos\theta - 3\sin\theta}$

c) if $t = \tan\frac{\theta}{2}$ then $\theta = 2\tan^{-1}t$ $\quad \frac{d\theta}{dt} = \frac{2}{1+t^2}$ so $d\theta = \frac{2dt}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$ $\quad \sin\theta = \frac{2t}{1+t^2}$

$$I = \int \frac{\frac{2dt}{1+t^2}}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} = \int \frac{2dt}{4(1-t^2) - 6t} = \int \frac{dt}{2(1-t^2) - 3t}$$

$-2t^2 - 3t + 2$. $\quad \Delta = 9 - 4 \times 2 \times (-2) = 25 = 5^2$ $\quad t_1 = \frac{3+5}{(-4)} = -2$ $\quad t_2 = 1/2$

So $\frac{1}{-2t^2 - 3t + 2} = \frac{a}{(t+2)} + \frac{b}{(-2t+1)}$

$\begin{cases} -2a + b = 0 \\ a + 2b = 1 \end{cases}$ $\quad \begin{cases} -2a + b = 0 \\ 2a + 4b = 2 \end{cases}$ $\Leftrightarrow$ $\begin{cases} 5b = 2 \\ a = \frac{b}{2} = \frac{1}{5} \end{cases}$ $\quad b = 2/5$

$$I = \int \frac{1/5\, dt}{t+2} + \frac{2}{5}\int \frac{dt}{-2t+1}$$

$$I = \frac{1}{5}\ln|t+2| - \frac{2}{5} \times \frac{1}{2}\ln|2t-1| + C$$

$$I = \frac{1}{5}\ln|t+2| - \frac{1}{5}\ln|2t-1| + C$$

$$I = \frac{1}{5}\ln\left|\frac{t+2}{2t-1}\right| + C = \frac{1}{5}\ln\left|\frac{\tan\theta/2 + 2}{2\tan\theta/2 - 1}\right| + C$$

18 Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{3}} \frac{1}{4+5\cos x}\,dx$.

$t = \tan\frac{x}{2}$ so $x = 2\tan^{-1}t$ $\quad \frac{dx}{dt} = \frac{2}{1+t^2}$ so $dx = \frac{2\,dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$ $\quad$ when $x = 0$, $t = \tan\frac{0}{2} = 0$

when $x = \pi/3$ $\quad t = \tan \pi/6 = \frac{1}{\sqrt{3}}$

$$\int_0^{\pi/3} \frac{dx}{4+5\cos x} = \int_0^{1/\sqrt{3}} \frac{\frac{2\,dt}{1+t^2}}{4+5\left(\frac{1-t^2}{1+t^2}\right)} = \int_0^{1/\sqrt{3}} \frac{2\,dt}{4+4t^2+5-5t^2}$$

$$\text{——} = \int_0^{1/\sqrt{3}} \frac{2\,dt}{9-t^2}$$

$$\frac{1}{9-t^2} = \frac{a}{3-t} + \frac{b}{3+t} \quad \text{so} \begin{cases} a-b=0 \\ 3a+3b=1 \end{cases} \quad \begin{cases} a=b \\ a=b=1/6 \end{cases}$$

$$I = 2\int_0^{1/\sqrt{3}} \frac{1/6}{3-t}\,dt + 2\int_0^{1/\sqrt{3}} \frac{1/6}{3+t}\,dt$$

$$I = \frac{1}{3}\left[\ln\left|\frac{3+t}{3-t}\right|\right]_0^{1/\sqrt{3}} = \frac{1}{3}\left[\ln\left|\frac{3+1/\sqrt{3}}{3-1/\sqrt{3}}\right| - \ln 1\right]$$

$$I = \frac{1}{3}\ln\left|\frac{3\sqrt{3}+1}{3\sqrt{3}-1}\right| = \frac{1}{3}\ln\left|\frac{(3\sqrt{3}+1)(3\sqrt{3}+1)}{(3\sqrt{3}-1)(3\sqrt{3}+1)}\right|$$

$$I = \frac{1}{3}\ln\left|\frac{27+6\sqrt{3}+1}{27-1}\right| = \frac{1}{3}\ln\left|\frac{28+6\sqrt{3}}{26}\right|$$

$$I = \frac{1}{3}\ln\left|\frac{14+3\sqrt{3}}{13}\right|$$

19 Find: $\int x \log_e 2x \, dx$ we integrate by parts.

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$u(x) = \log_e 2x$ $u'(x) = \frac{1}{2x} \times 2 = \frac{1}{x}$

$v(x) = \frac{x^2}{2}$ $v'(x) = x$

$$\int x \ln 2x \, dx = \left[\frac{x^2 \ln 2x}{2} \right] - \int \frac{1}{x} \times \frac{x^2}{2} \, dx + C$$

$$\int x \ln 2x \, dx = \frac{x^2 \ln 2x}{2} - \frac{1}{2} \int x \, dx + C$$

$$\int x \ln 2x \, dx = \frac{x^2 \ln 2x}{2} - \frac{x^2}{4} + C$$

$$\int x \ln 2x \, dx = \frac{x^2}{4} \left[2 \ln 2x - 1 \right] + C$$

20 Find: **(a)** $\int \frac{x^2+1}{x^3+3x}dx$ **(b)** $\int \frac{dx}{x^2+2x+1}$

a) $\frac{x^2+1}{x^3+3x} = \frac{a}{x} + \frac{bx+c}{x^2+3}$ $\begin{cases} a+b=1 \\ 3a=1 \end{cases}$ $\begin{cases} a=1/3 \\ b=2/3 \\ c=0 \end{cases}$

$$\int \frac{x^2+1}{x^3+3x}dx = \int \frac{1/3}{x}dx + \int \frac{2/3\,x}{x^2+3}dx$$

$$\text{———} = \frac{1}{3}\int \frac{dx}{x} + \frac{1}{3}\int \frac{2x}{x^2+3}dx$$

$$\text{———} = \frac{1}{3}\ln|x| + \frac{1}{3}\ln|x^2+3| + C = \frac{1}{3}\ln|x^3+3x| + C$$

b) $\int \frac{dx}{x^2+2x+1} = \int \frac{dx}{(x+1)^2}$

we do a change of variable

$u = x+1$ $\frac{du}{dx} = 1$ $du = dx$

$$\text{———} = \int \frac{du}{u^2} = \int u^{-2}du$$

$$\text{———} = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + c$$

$$\text{———} = -\frac{1}{x+1} + C$$

20 Find: (c) $\int \frac{x^3+1}{x}dx$ (d) $\int \frac{x+1}{\sqrt{x^2+2x-3}}dx$

c) $\int \frac{x^3+1}{x}\,dx = \int x^2 + \frac{1}{x}\,dx = \frac{x^3}{3} + \ln|x| + C$

d) $\int \frac{x+1}{\sqrt{x^2+2x-3}}\,dx = \int \frac{x+1}{\sqrt{(x-1)(x+3)}}\,dx$

we do a change of variable $u(x) = x^2 + 2x - 3$

$\frac{du}{dx} = 2x + 2$ so $(x+1)\,dx = \frac{du}{2}$

$I = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot\int u^{-1/2}\,du = \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = u^{1/2} + C$

$\therefore \int \frac{x+1}{\sqrt{x^2+2x-3}}\,dx = \sqrt{x^2+2x-3} + C$

20 Find: (e) $\int \frac{x+4}{x^3+4x}dx$ (f) $\int \frac{dx}{x^3-1}$

e) $\frac{x+4}{x^3+4x} = \frac{x+4}{x(x^2+4)} = \frac{a}{x} + \frac{bx+c}{x^2+4}$

So $\begin{cases} a+b=0 \\ c=1 \\ 4a=4 \end{cases}$ $\begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases}$ $\frac{x+4}{x^3+4x} = \frac{1}{x} + \frac{(-x)+1}{x^2+4}$

$\int \frac{x+4}{x^3+4x}dx = \int \frac{dx}{x} - \int \frac{x-1}{x^2+4}dx$

$\_\_\_ = \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$

$\_\_\_ = \ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$\_\_\_ = \ln\left|\frac{x}{\sqrt{x^2+4}}\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

f) $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$

$\Delta = 1-4\times1 = -3 < 0$
so no roots for x^2+x+1

$\_\_\_ = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$

$\begin{cases} a+b=0 \\ a-b+c=0 \\ a-c=1 \end{cases}$ $\begin{cases} c=a-1 \\ a+b=0 \\ 2a-b=1 \end{cases}$

$a=1/3 \quad b=-1/3 \quad c=-2/3$

$\_\_\_ = \frac{1/3}{x-1} + \frac{(-1/3)x-2/3}{x^2+x+1}$

$I = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x-2}{x^2+x+1}dx + C$

$I = \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x-4}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\left[\int \frac{2x+1-5}{x^2+x+1}dx\right]$

$I = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| + \frac{5}{6}\int \frac{dx}{(x+\frac{1}{2})^2 - \frac{1}{4}+1}$

$\frac{5}{6}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$

$I = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| + \frac{5}{6}\int \frac{dx}{(x+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{5}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

20 Find: **(g)** $\int x \sin^{-1} x \, dx$

by parts

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$u(x) = \sin^{-1} x \qquad u'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$v(x) = \dfrac{x^2}{2} \qquad v'(x) = x$

$$\int x \sin^{-1} x \, dx = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx$$

we do a change of variable $x = \sin\theta \qquad \dfrac{dx}{d\theta} = \cos\theta \qquad dx = \cos\theta \, d\theta$

$$\int x \sin^{-1} x \, dx = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2} \int \frac{\sin^2\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta} \, d\theta$$

$\cos 2x = 1 - 2\sin^2 x$

so $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$$I = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \int 1 - \cos 2\theta \, d\theta$$

$$I = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \left(\theta - \frac{\sin 2\theta}{2} \right) + C$$

$$I = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \sin^{-1} x + \frac{1}{8} \underbrace{\sin 2(\sin^{-1} x)}_{2 \sin(\sin^{-1} x) \cos(\sin^{-1} x)} + C$$

$$I = \frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{4} x \sqrt{1-x^2} + C$$

$$\therefore \int x \sin^{-1} x \, dx = \frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{4} x \sqrt{1-x^2} + C$$

21 Find: (a) $\int \frac{dx}{x^2 - 4x - 1}$ (b) $\int \frac{dx}{3x^2 + 6x + 10}$

a) $\Delta = 4^2 - 4 \times (-1) = 20 = (2\sqrt{5})^2$ $x_1 = \frac{4 + 2\sqrt{5}}{2} = 2 + \sqrt{5}$ $x_2 = 2 - \sqrt{5}$

$$\frac{1}{x^2 - 4x - 1} = \frac{a}{(x - (2+\sqrt{5}))} + \frac{b}{(x - (2 - \sqrt{5}))}$$

$\begin{cases} a + b = 0 \\ -2b - \sqrt{5}b - 2a + \sqrt{5}a = 1 \end{cases}$ $\begin{cases} b = -a \\ \cancel{2a} + \sqrt{5}a - \cancel{2a} + \sqrt{5}a = 1 \end{cases}$ $\begin{cases} a = 1/2\sqrt{5} \\ b = -1/2\sqrt{5} \end{cases}$

$$\int \frac{dx}{x^2 - 4x - 1} = \frac{1}{2\sqrt{5}} \left[\int \frac{dx}{[x - (2+\sqrt{5})]} - \int \frac{dx}{[x - (2 - \sqrt{5})]} \right]$$

$$\text{———} = \frac{1}{2\sqrt{5}} \ln \left[\frac{x - (2+\sqrt{5})}{x - (2 - \sqrt{5})} \right] + C$$

b) $\Delta = 6^2 - 4 \times 10 \times 3 = -84 < 0$ so no roots.

we complete the square.

$$I = \frac{1}{3} \int \frac{dx}{x^2 + 2x + 10/3} = \frac{1}{3} \int \frac{dx}{(x+1)^2 - 1 + \frac{10}{3}} = \frac{1}{3} \int \frac{dx}{(x+1)^2 + 7/3}$$

$$I = \frac{1}{3} \times \frac{1}{\left(\sqrt{\frac{7}{3}}\right)} \tan^{-1} \left(\frac{x+1}{\sqrt{\frac{7}{3}}} \right) + C$$

$$I = \frac{1}{\sqrt{21}} \tan^{-1} \left[\frac{\sqrt{3}(x+1)}{\sqrt{7}} \right] + C$$

21 Find: (c) $\int \frac{dx}{\sqrt{x^2 - 4x + 1}}$ (d) $\int \frac{dx}{\sqrt{x^2 + 16}}$

c) we know that $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

$$\int \frac{dx}{\sqrt{x^2-4x+1}} = \int \frac{dx}{\sqrt{(x-2)^2-4+1}} = \int \frac{dx}{\sqrt{(x-2)^2-3}}$$

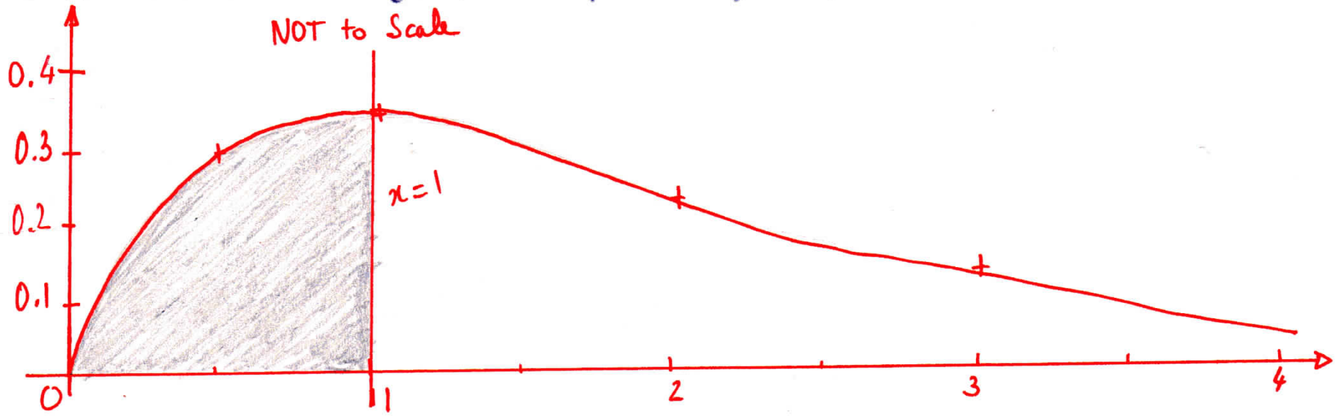
$u = x-2 \qquad \frac{du}{dx} = 1 \qquad du = dx \qquad = \int \frac{du}{\sqrt{u^2-(\sqrt{3})^2}}$

$$= \ln\left|u+\sqrt{u^2-3}\right| + C = \ln\left|x-2+\sqrt{x^2-4x+1}\right| + C$$

d) we know that $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

$$\therefore \int \frac{dx}{\sqrt{x^2+16}} = \int \frac{dx}{\sqrt{x^2+4^2}} = \ln\left|x+\sqrt{x^2+16}\right| + C$$

23 Calculate the area of the region bounded by the curve $y = xe^{-x}$, the x-axis and the line $x = 1$.



This area is $\int_0^1 x e^{-x}\, dx$ - we integrate by parts.

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$u(x) = x \qquad u'(x) = 1$

$v(x) = -e^{-x} \qquad v'(x) = e^{-x}$

$$I = \left[x \times (-e^{-x})\right]_0^1 - \int_0^1 1 \times (-e^{-x})\, dx$$

$$I = \left[xe^{-x}\right]_1^0 + \int_0^1 e^{-x}\, dx$$

$$I = \left[0e^{-0} - 1e^{-1}\right] + \left[-e^{-x}\right]_0^1$$

$$I = -\frac{1}{e} + \left[e^{-x}\right]_1^0$$

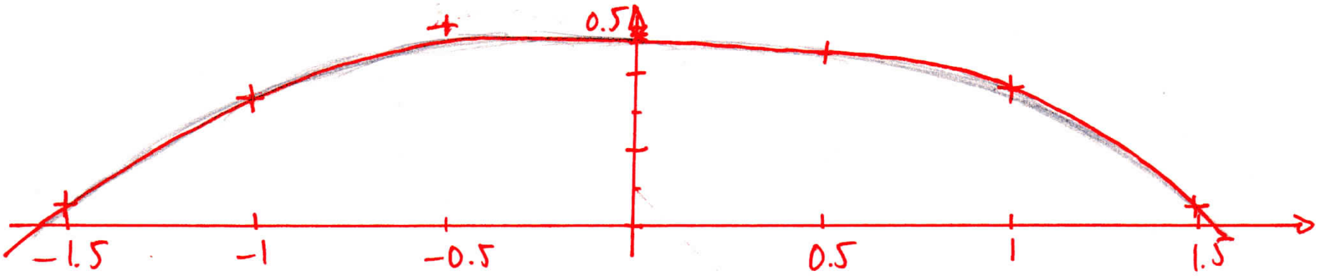
$$I = -\frac{1}{e} + e^0 - e^{-1}$$

$$I = 1 - \frac{2}{e} \simeq 0.26 \text{ approx}$$

FURTHER INTEGRATION - CHAPTER REVIEW

24 Sketch the graph of $y = \dfrac{\cos x}{1 + \cos x}$ for $-\pi < x < \pi$, stating the coordinates of its intersection with the x-axis and of the turning point. Find the area of the region bounded by the curve and the x-axis.

$y = 0$ when $\cos x = 0$, i.e. $x = \pm \pi/2$



$$\text{Area} = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \cos x}\, dx$$ we use the t-formulae

if $t = \tan \dfrac{x}{2}$ then $\cos x = \dfrac{1-t^2}{1+t^2}$ $\qquad x = 2\tan^{-1} t$, $\quad \dfrac{dx}{dt} = \dfrac{2}{1+t^2}$

$$\text{Area} = \int_{-1}^{1} \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \times \frac{2\, dt}{1+t^2}$$

$$\text{Area} = 2\int_{-1}^{1} \frac{1-t^2}{1+\cancel{t^2}+1-\cancel{t^2}} \times \frac{dt}{1+t^2} = \int_{-1}^{1} \frac{1-t^2}{1+t^2}\, dt$$

$$\text{Area} = \int_{1}^{-1} \frac{t^2-1}{t^2+1}\, dt = \int_{1}^{-1} \frac{t^2+1-2}{t^2+1}\, dt = \int_{1}^{-1} dt - 2\int_{1}^{-1} \frac{dt}{1+t^2}$$

$$\text{Area} = \Big[t\Big]_{1}^{-1} - 2\Big[\tan^{-1} t\Big]_{1}^{-1}$$

$$\text{Area} = -1-1 - 2\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$\text{Area} = -2 + 2 \times \frac{\pi}{4} - 2\left(\frac{-\pi}{4}\right) = \pi - 2 \text{ units}^2$$

27 **(a)** Using the substitution $u = a - x$, or otherwise, prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

(b) Hence evaluate $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$.

a) $I = \int_0^a f(a-x)\, dx$ $\quad u = a - x$ so $\frac{du}{dx} = -1$ so $du = -dx$

$dx = -du$

when $x = 0$ $\quad u = a$

$x = a$ $\quad u = 0$

$$I = \int_a^0 f(u) \times (-du) = -\int_a^0 f(u)\, du = \int_0^a f(u)\, du$$

$$\therefore \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

b) $I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+(\cos(\pi-x))^2} dx$

$$I = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x} dx = \int_0^\pi \frac{\pi \sin x\, dx}{1+\cos^2 x} - I.$$

$\therefore 2I = \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x} dx$ $\quad$ we do a change of variable

$u = \cos x$ so $\frac{du}{dx} = -\sin x$

so $dx \sin x = -du$

$$I = \frac{\pi}{2} \int_1^{-1} \frac{-du}{1+u^2}$$

$$I = \frac{\pi}{2} \int_{-1}^{1} \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi}{2}\left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

$$I = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{2} \times \frac{\pi}{2} = \frac{\pi^2}{4}$$