TRIGONOMETRIC EQUATIONS - CHAPTER REVIEW

1 (a) Express $2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)-2\cos\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

(b) Hence, or otherwise, solve $2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)-2\cos\theta=1$ for $0<\theta<2\pi$.

a) $2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)-2\cos\theta = 2\sqrt{3}\left[\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}\right]-2\cos\theta$

$= 2\sqrt{3}\left[\cos\theta\times\frac{\sqrt{3}}{2}-\sin\theta\sin\frac{\pi}{6}\right]-2\cos\theta$

$= 3\cos\theta-\sqrt{3}\sin\theta-2\cos\theta = \cos\theta-\sqrt{3}\sin\theta$

$= 2\left[\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right]$

$= 2\left[\cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta\right]$

$= 2\cos\left(\theta+\frac{\pi}{3}\right)$

b) So $2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)-2\cos\theta=1$

$\iff 2\cos\left(\theta+\frac{\pi}{3}\right)=1$

$\iff \cos\left(\theta+\frac{\pi}{3}\right)=\frac{1}{2}=\cos\frac{\pi}{3}$

so the general solution is $\theta+\frac{\pi}{3}=\pm\frac{\pi}{3}+2n\pi$

$\theta=\pm\frac{\pi}{3}-\frac{\pi}{3}+2n\pi$

$n=0$ gives $\theta=0$ (outside of $(0,2\pi)$) and $\theta=-\frac{2\pi}{3}$ (also outside)

$n=1$ gives $\theta=2\pi$ (outside of $(0,2\pi)$) and $\theta=\frac{4\pi}{3}$

$n=2$ gives $\theta=4\pi-\frac{2\pi}{3}=\frac{10\pi}{3}$ (outside of $(0,2\pi)$)

all other solutions are outside $(0,2\pi)$

So only one solution: $\theta=\frac{4\pi}{3}$

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2 **(a)** Express $3\sin x + 4\cos x$ in the form $r\sin(x+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$.

(b) Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \le x \le 2\pi$. Give answer(s) to two decimal places.

(c) Write the general solution for $3\sin x + 4\cos x = 5$.

a) $3\sin x + 4\cos x = 5\left[\frac{3}{5}\sin x + \frac{4}{5}\cos x\right]$ $\qquad 3^2+4^2=25$

$= 5\left[\cos\alpha \sin x + \sin\alpha \cos x\right]$ where $\begin{cases} \cos\alpha = 3/5 \\ \sin\alpha = 4/5 \end{cases}$

$= 5\sin(\alpha + x)$. $\qquad \alpha \simeq 0.927$ rad.

b) $3\sin x + 4\cos x = 5 \iff 5\sin(\alpha+x) = 5$

$\iff \sin(\alpha + x) = 1$

$\iff \alpha + x = (-1)^n\frac{\pi}{2} + n\pi$

so $x = (-1)^n\frac{\pi}{2} - \alpha + n\pi$

$x \simeq 0.64$

c) $x = (-1)^n\frac{\pi}{2} - \alpha + n\pi$

3 Use $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$ to find the exact value of $\tan\frac{\pi}{8}$.

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos(2\times\pi/8)}{1+\cos(2\times\pi/8)}}$$

$$\_\_\_\_ = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}}$$

$$\_\_\_\_ = \sqrt{\frac{1-\sqrt{2}/2}{1+\sqrt{2}/2}}$$

$$\_\_\_\_ = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$\_\_\_\_ = \sqrt{\frac{(2-\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}}$$

we rationalise the denominator

$$\_\_\_\_ = \sqrt{\frac{4+2-4\sqrt{2}}{4-2}}$$

$$\_\_\_\_ = \sqrt{\frac{6-4\sqrt{2}}{2}}$$

$$\_\_\_\_ = \sqrt{3-2\sqrt{2}}$$

4. First show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ [by expanding $\cos 3\theta$ as $\cos(2\theta + \theta)$]

$\cos 3\theta = \cos(2\theta + \theta)$

$\quad = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$\quad = (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta \times \sin\theta$

$\quad = 2\cos^3\theta - \cos\theta - 2\cos\theta \times \sin^2\theta$

$\quad = 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$\quad = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Show that the cubic equation $8x^3 - 6x + 1 = 0$ can be reduced to the form $\cos 3\theta = \frac{-1}{2}$ by substituting $x = \cos\theta$, and then solve this equation.

Substituting $x = \cos\theta$ in the cubic equation results in:

$8\cos^3\theta - 6\cos\theta + 1 = 0$

$\Leftrightarrow 2(4\cos^3\theta - 3\cos\theta) + 1 = 0$

$\Leftrightarrow 2 \times \cos 3\theta + 1 = 0 \Leftrightarrow \cos 3\theta = -1/2$

The general solution of this equation is:

$3\theta = \pm\frac{2\pi}{3} + 2n\pi$ or $\theta = \pm\frac{2\pi}{9} + \frac{2n\pi}{3}$

$n = 0$ gives $\theta = \pm\frac{2\pi}{9}$

$n = 1$ gives $\theta = \frac{8\pi}{9}$ and $\theta = \frac{4\pi}{9}$

So the 3 solutions of the cubic (which has only 3 solutions) are: $\cos\frac{2\pi}{9}$, $\cos\frac{8\pi}{9}$ and $\cos\frac{4\pi}{9}$

(these solutions are all different)

From this, deduce the following:

a) $\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=0$

b) $\sec\left(\frac{2\pi}{9}\right)\sec\left(\frac{4\pi}{9}\right)\sec\left(\frac{8\pi}{9}\right)=-8$

c) $\sec\left(\frac{2\pi}{9}\right)+\sec\left(\frac{4\pi}{9}\right)+\sec\left(\frac{8\pi}{9}\right)=6$

d) $\tan^2\left(\frac{2\pi}{9}\right)+\tan^2\left(\frac{4\pi}{9}\right)+\tan^2\left(\frac{8\pi}{9}\right)=33$

let $\alpha=\cos\frac{2\pi}{9}$, $\beta=\cos\frac{4\pi}{9}$, $\gamma=\cos\frac{8\pi}{9}$.

α,β and γ are solutions of the cubic equation $8x^3-6x+1=0$

a) $\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=\alpha+\beta+\gamma=-\frac{b}{a}=0$

b) $\sec\left(\frac{2\pi}{9}\right)\sec\left(\frac{4\pi}{9}\right)\sec\left(\frac{8\pi}{9}\right)=\frac{1}{\alpha}\times\frac{1}{\beta}*\frac{1}{\gamma}=\frac{1}{\alpha\beta\gamma}=\frac{1}{-d/a}=-\frac{a}{d}=\frac{-8}{1}=-8$

c) $\sec\left(\frac{2\pi}{9}\right)+\sec\left(\frac{4\pi}{9}\right)+\sec\left(\frac{8\pi}{9}\right)=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\beta\alpha+\alpha\gamma}{\alpha\beta\gamma}=\frac{c/a}{-d/a}=-\frac{c}{d}=\frac{6}{1}=6$

d) $\tan^2\left(\frac{2\pi}{9}\right)+\tan^2\left(\frac{4\pi}{9}\right)+\tan^2\left(\frac{8\pi}{9}\right)=\left[\sec^2\left(\frac{2\pi}{9}\right)-1\right]+\left[\sec^2\left(\frac{4\pi}{9}\right)-1\right]+\left[\sec^2\left(\frac{8\pi}{9}\right)-1\right]$

$$\_\_\_=\left[\frac{1}{\alpha^2}-1\right]+\left[\frac{1}{\beta^2}-1\right]+\left[\frac{1}{\gamma^2}-1\right]=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}-3$$

$$\_\_\_=\frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\gamma^2}{\alpha^2\beta^2\gamma^2}-3$$

We know that $(a+b+c)^2=a^2+b^2+c^2-2(ab+ac+bc)$, therefore

$$\_\_\_=\frac{(\beta\gamma+\alpha\gamma+\alpha\gamma)^2-2(\alpha\beta\gamma^2+\alpha\gamma\beta^2+\beta\gamma\alpha^2)}{(\alpha\beta\gamma)^2}-3$$

$$\_\_\_=\frac{(c/a)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(-d/a)^2}-3$$

$$\_\_\_=\frac{(-6/8)^2-2\times(-1/8)\times(0)}{(-1/8)^2}-3$$

$$\_\_\_=\frac{36/64}{1/64}-3=36-3=33$$

5 It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Use this result to solve $\cos 3\theta + \cos 2\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$.

$$\cos 3\theta + \cos 2\theta + \cos\theta = 0 \iff 4\cos^3\theta - 3\cos\theta + 2\cos^2\theta - 1 + \cos\theta = 0$$

$$\iff 4\cos^3\theta + 2\cos^2\theta - 2\cos\theta - 1 = 0$$

$$\iff 2\cos^2\theta\,[2\cos\theta + 1] - [2\cos\theta + 1] = 0$$

$$\iff [2\cos\theta + 1][2\cos^2\theta - 1] = 0$$

So either $2\cos\theta + 1 = 0$ or $2\cos^2\theta = 1$

Case ① $\cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\theta = \pm\frac{2\pi}{3} + 2n\pi$ Solutions are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

Case ② $\cos^2\theta = \frac{1}{2} \iff \cos\theta = \pm\frac{\sqrt{2}}{2} = \begin{cases} \cos\frac{\pi}{4} \\ \cos 3\pi/4 \end{cases}$

$\theta = \pm\frac{\pi}{4} + 2n\pi$ Solutions are $\frac{\pi}{4}$, $\frac{7\pi}{4}$

$\theta = \pm\frac{3\pi}{4} + 2n\pi$ Solutions are $\frac{3\pi}{4}$, $\frac{5\pi}{4}$

So the solutions are: $\frac{\pi}{4}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{4\pi}{3}$, $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$

6 (a) Expand $\cos(2A+B)$ and hence prove that $\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$.

(b) By writing $x = k\cos\theta$ and giving k a suitable value, use the formula proved in part (a) to find the three roots of the equation $27x^3 - 9x = 1$. Hence write the value of the product $\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$.

a) $\cos(2A+B) = \cos 2A \cos B - \sin 2A \sin B$

$= (2\cos^2 A - 1)\cos B - 2\sin A\cos A \times \sin B$

$= 2\cos^2 A\cos B - \cos B - 2\sin A\cos A\sin B.$

if $B = A$, then: $\cos 3A = 2\cos^2 A\cos A - \cos A - 2\sin^2 A\cos A$

$\Leftrightarrow \cos 3A = 2\cos^3 A - \cos A - 2\cos A(1-\cos^2 A)$

$\Leftrightarrow \cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \frac{1}{4}\cos 3A = \cos^3 A - \frac{3}{4}\cos A$

b) $27x^3 - 9x = 1 \Leftrightarrow 27(k\cos\theta)^3 - 9\times k\cos\theta = 1$

$\Leftrightarrow 9[3k^3\cos^3\theta - k\cos\theta] = 1$

$\Leftrightarrow 9k[3k^2\cos^3\theta - \cos\theta] = 1$

if we pick k such that $3k^2 = \frac{4}{3}$, i.e. $k^2 = \frac{4}{9}$

$\Rightarrow k = 2/3$

then $\Rightarrow 9\times\frac{2}{3}\left[\frac{4}{3}\cos^3\theta - \cos\theta\right] = 1$

$\Rightarrow 6\left[\frac{1}{3}\cos 3\theta\right] = 1 \Leftrightarrow \cos 3\theta = 1/2 = \cos\frac{\pi}{3}$

$3\theta = \pm\frac{\pi}{3} + 2n\pi \Rightarrow \theta = \pm\frac{\pi}{9} + \frac{2n\pi}{3}$

Solutions are $\pi/9$, $\frac{7\pi}{9}$, $\frac{5\pi}{9}$ so solutions are $\frac{2}{3}\cos\frac{\pi}{9}, \frac{2}{3}\cos\frac{5\pi}{9}, \frac{2}{3}\cos\frac{7\pi}{9}$

Product of roots: $\frac{2}{3}\cos\frac{\pi}{9}\times\frac{2}{3}\cos\frac{5\pi}{9}\times\frac{2}{3}\cos\frac{7\pi}{9} = -\frac{d}{a} = \frac{1}{27}$

so $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$

and $\therefore \cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}\times\cos\frac{\pi}{3} = \frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$

7 If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation $x^3 - (a+1)x^2 + (c-a)x - c = 0$, show that $\alpha+\beta+\gamma = n\pi + \frac{\pi}{4}$.

$$\tan\alpha + \tan\beta + \tan\gamma = a+1$$

$$\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma = c - a$$

$$\tan\alpha\tan\beta\tan\gamma = c$$

So $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan(\beta+\gamma)}{1 - \tan\alpha\tan(\beta+\gamma)}$$

$$= \frac{\tan\alpha + \left[\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}\right]}{1 - \tan\alpha \times \left[\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}\right]}$$

$$= \frac{\tan\alpha\left[1-\tan\beta\tan\gamma\right] + \tan\beta + \tan\gamma}{\left[1-\tan\beta\tan\gamma\right] - \tan\alpha(\tan\beta+\tan\gamma)}$$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\alpha\tan\beta - \tan\alpha\tan\gamma}$$

$$= \frac{a+1-c}{1-(c-a)} = \frac{a+1-c}{a+1-c} = 1$$

$$\therefore \tan(\alpha+\beta+\gamma) = 1 = \tan\pi/4$$

$$\therefore \alpha+\beta+\gamma = \frac{\pi}{4} + n\pi$$

8 Solve $\sin x = \cos 5x$ for $0 < x < \pi$. $\iff \cos\left(\frac{\pi}{2} - x\right) = \cos 5x$

General solution is $5x = \pm\left(\frac{\pi}{2} - x\right) + 2n\pi$

$$\iff \begin{cases} 5x - x = -\frac{\pi}{2} + 2n\pi \\ 5x + x = +\frac{\pi}{2} + 2n\pi \end{cases} \iff \begin{cases} 4x = -\frac{\pi}{2} + 2n\pi \\ 6x = \frac{\pi}{2} + 2n\pi \end{cases}$$

$$\iff \begin{cases} x = -\frac{\pi}{8} + \frac{n\pi}{2} & \text{equation } ① \\ x = \frac{\pi}{12} + \frac{n\pi}{3} & \text{equation } ② \end{cases}$$

① : $n = 1$ gives $x = \frac{3\pi}{8}$

$n = 2$ gives $x = \frac{7\pi}{8}$

② $n = 0$ gives $x = \frac{\pi}{12}$

$n = 1$ gives $x = \frac{5\pi}{12}$

$n = 2$ gives $x = \frac{3\pi}{4}$

So 5 solutions: $\frac{\pi}{12}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{8}$

9 **(a)** Find A and B in terms of x and y such that $\sin x + \sin y = 2\sin A\cos B$.

(b) Find the solution of $\sin\theta + \sin 2\theta + \sin 3\theta = 0$ for $0 \le \theta \le \pi$.

a) From $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\therefore \sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\therefore \begin{cases} x = A+B \\ y = A-B \end{cases} \Rightarrow \begin{cases} 2A = x+y \\ 2B = x-y \end{cases} \Rightarrow \begin{cases} A = \frac{x+y}{2} \\ B = \frac{x-y}{2} \end{cases}$

b) $\sin\theta + \sin 2\theta + \sin 3\theta = 0$

$\Leftrightarrow 2\sin\left(\frac{\theta+2\theta}{2}\right)\cos\left(\frac{\theta-2\theta}{2}\right) + \sin 3\theta = 0$

$\Leftrightarrow 2\sin\frac{3\theta}{2}\cos\frac{\theta}{2} + 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2} = 0$

$\Leftrightarrow \sin\frac{3\theta}{2}\left[\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right] = 0.$

So either $\sin\frac{3\theta}{2} = 0$, i.e. $\frac{3\theta}{2} = 2n\pi$ $\theta = \frac{4n\pi}{3}$

$\theta = 0$, $\theta = \frac{4\pi}{3}$

OR $\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} = 0$

$\Leftrightarrow \cos\frac{3\theta}{2} = -\cos\frac{\theta}{2} = \cos\left(\pi - \frac{\theta}{2}\right)$,

General solution is $\frac{3\theta}{2} = \left(\pi - \frac{\theta}{2}\right) + 2n\pi$ or $\frac{3\theta}{2} = -\left(\pi - \frac{\theta}{2}\right) + 2n\pi$

i.e. $2\theta = \pi + 2n\pi$ OR $\theta = -\pi + 2n\pi$

$\theta = \frac{\pi}{2} + n\pi$ $\theta = \pi$

$\theta = \frac{\pi}{2}$

So 4 solutions: $0, \frac{\pi}{2}, \frac{4\pi}{3}, \pi$