Example 6

Solve the equation $2\sin\left(x+\frac{5\pi}{6}\right)=\sin x$, for $0 \le x \le 2\pi$.

Solution

$$2\left(\sin x\cos\frac{5\pi}{6}+\cos x\sin\frac{5\pi}{6}\right)=\sin x$$

$$2\sin x\times\left(-\frac{\sqrt{3}}{2}\right)+2\cos x\times\frac{1}{2}=\sin x$$

$$-\sqrt{3}\sin x+\cos x=\sin x$$

$$(1+\sqrt{3})\sin x=\cos x$$

$$\tan x=\frac{1}{\sqrt{3}+1}$$

$$x=0.3509,\ \pi+0.3509$$

$$x=0.351,\ 3.493$$

Example 7

Solve for $0 \le x \le 2\pi$: **(a)** $4\cos x=\operatorname{cosec} x$ **(b)** $\cos 4x-\cos 2x=0$

Solution

(a)

$$4\cos x=\operatorname{cosec} x$$

$$4\cos x=\frac{1}{\sin x}$$

$$4\sin x\cos x=1$$

$$2\sin 2x=1$$

$$\sin 2x=0.5$$

$$2x=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$$

$$x=\frac{\pi}{12},\frac{5\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12}$$

(b)

$$\cos 4x-\cos 2x=0$$

$$2\cos^2 2x-1-\cos 2x=0$$

$$2\cos^2 2x-\cos 2x-1=0$$

$$(\cos 2x-1)(2\cos 2x+1)=0$$

$$\cos 2x=1,\ \cos 2x=-0.5$$

$$2x=0,2\pi,4\pi \quad \textbf{or} \quad 2x=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\frac{10\pi}{3}$$

$$x=0,\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{5\pi}{3},2\pi$$

This equation could also have been solved by writing $\cos 4x=\cos 2x$.

Example 9

(a) Use the expansion of $\sin(2\theta+\theta)$ to obtain an expression for $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence find the roots of $4x^3-3x+0.5=0$.

Solution

(a)

$$\begin{aligned}\sin 3\theta&=\sin(2\theta+\theta)\\&=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\&=2\sin\theta\cos\theta\cos\theta+(1-2\sin^2\theta)\sin\theta\\&=2\sin\theta\cos^2\theta+\sin\theta-2\sin^3\theta\\&=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta\\&=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta\end{aligned}$$

$$\sin 3\theta=3\sin\theta-4\sin^3\theta$$

(b) Let $x=\sin\theta$: $4\sin^3\theta-3\sin\theta+0.5=0$

$$3\sin\theta-4\sin^3\theta=0.5$$

$$\sin 3\theta=\frac{1}{2}$$

$$3\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$$

$$\theta=\frac{\pi}{18},\frac{5\pi}{18},\frac{13\pi}{18},\frac{17\pi}{18},\frac{25\pi}{18},\frac{29\pi}{18}$$

Hence the roots are $x=\sin\frac{\pi}{18},\ \sin\frac{5\pi}{18},\ \sin\frac{25\pi}{18}$.

Example 8

Solve for $0 \le x \le 2\pi$: **(a)** $\cos 3x = \cos 2x \cos x$ **(b)** $\sin 7x - \sin x = \sin 3x$

Solution

(a)

$$\cos 3x = \cos 2x \cos x$$
$$\cos 3x = \frac{1}{2}(\cos 3x + \cos x)$$
$$2\cos 3x = \cos 3x + \cos x$$
$$\cos 3x = \cos x$$
$$3x = x,\ 2\pi - x,\ 2\pi + x,\ 4\pi - x,\ 4\pi + x,\ 6\pi - x,\ 6\pi + x,\ 8\pi - x$$
$$2x = 0,\ 2\pi,\ 4\pi,\ 6\pi \quad \textbf{or} \quad 4x = 2\pi,\ 4\pi,\ 6\pi,\ 8\pi$$
$$x = 0,\ \pi,\ 2\pi \quad \textbf{or} \quad x = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$$

The solution is $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(b)

$$\sin 7x - \sin x = \sin 3x$$
$$2\cos 4x \sin 3x = \sin 3x$$
$$\sin 3x(2\cos 4x - 1) = 0$$
$$\sin 3x = 0,\ \cos 4x = 0.5$$
$$3x = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi \quad \textbf{or} \quad 4x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3}$$
$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi \quad \textbf{or} \quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$
$$x = 0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{5\pi}{3}, 2\pi$$

Example 10

Solve $5\cos\theta - 2\sin\theta = 2$ for $0° \le \theta \le 360°$ using the t formulae.

Solution

$t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\tan\theta = \frac{2t}{1-t^2}$.

Substitute into the equation: $5 \times \frac{1-t^2}{1+t^2} - 2 \times \frac{2t}{1+t^2} = 2$

Simplify: $5 - 5t^2 - 4t = 2 + 2t^2$

$$7t^2 + 4t - 3 = 0$$
$$(7t - 3)(t + 1) = 0$$
$$t = \frac{3}{7}, -1$$

$0° \le \frac{\theta}{2} \le 180°$: $\frac{\theta}{2} = 23°\ 12', 180° - 45°$

$$\theta = 46°\ 24', 270°.$$

Test whether $\theta = 180°$ is a solution: LHS $= 5\cos 180° - 2\sin 180° = -5 - 0 \ne 2$

This equation could also have been solved using the auxiliary angle method.

Example 11

(a) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.

(b) By using suitable substitutions for A and B, show that $\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

(c) Hence solve $\sin 2x+\sin 4x=\sin 6x$ for $0\le x\le\pi$.

Solution

(a) $\text{LHS}=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$

$=2\sin A\cos B$

(b) Let $x=A+B$ and $y=A-B$.

Adding these equations: $A=\frac{x+y}{2}$ Subtracting the equations: $B=\frac{x-y}{2}$

Substitute these results in **(a)**: $\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(c) Use the result in **(b)** on the LHS: $\sin 2x+\sin 4x=2\sin 3x\cos(-x)=2\sin 3x\cos x$

Use the double angle formula $\sin 2x=2\sin x\cos x$ on the RHS: $\sin 6x=2\sin 3x\cos 3x$

$2\sin 3x\cos(-x)=2\sin 3x\cos 3x$

$2\sin 3x\cos x=2\sin 3x\cos 3x$ ($\cos x$ is an even function, so $\cos(-x)=\cos x$)

$2\sin 3x(\cos x-\cos 3x)=0$

$\sin 3x=0$ **or** $\cos x-\cos 3x=0$

$3x=0,\pi,2\pi,3\pi$ **or** $\cos 3x=\cos x$

$x=0,\frac{\pi}{3},\frac{2\pi}{3},\pi$ **or** $3x=x,2\pi-x,2\pi+x,4\pi-x$

$2x=0,4x=2\pi,2x=2\pi,4x=4\pi$

$x=0,\frac{\pi}{2},\pi$

Solution is $x=0,\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\pi$.

Example 12

Solve the equation $\cos 4x+\sin 3x=0$ for $0\le x\le\pi$.

Solution

Rewrite equation: $\cos 4x=-\sin 3x$

Sine is an odd function, so: $-\sin 3x=\sin(-3x)$: $\cos 4x=\sin(-3x)$

Use $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ to rewrite equation: $\cos 4x=\cos\left(\frac{\pi}{2}+3x\right)$

$$4x=\frac{\pi}{2}+3x,\ 2\pi-\left(\frac{\pi}{2}+3x\right),\ 2\pi+\left(\frac{\pi}{2}+3x\right),\ 4\pi-\left(\frac{\pi}{2}+3x\right),\ 6\pi-\left(\frac{\pi}{2}+3x\right),\ 8\pi-\left(\frac{\pi}{2}+3x\right)$$

$$x=\frac{\pi}{2},\ 7x=\frac{3\pi}{2},\ x=\frac{5\pi}{2},\ 7x=\frac{7\pi}{2},\frac{11\pi}{2},\frac{15\pi}{2}.$$

$$x=\frac{\pi}{2},\frac{3\pi}{14},\frac{\pi}{2},\frac{11\pi}{14},\frac{15\pi}{14}.$$

As $0\le x\le\pi$, the solution is $x=\frac{3\pi}{14},\frac{\pi}{2}$ **or** $\frac{11\pi}{14}$.