COMPLEX NUMBERS AND POLYNOMIAL EQUATIONS

1 Use the factor theorem to show that:

(a) $z-i$ is a factor of z^3+2iz^2+3i **(b)** $z-3$ is a factor of $z^2-(5-i)z+6-3i$

(c) $z+2-i$ is a factor of $2z^3+3z^2-(5+2i)z-17-9i$

(d) $z-3+\sqrt{2}i$ is a factor of $2z^4-12z^3+23z^2-6z+11$.

a) $i^3+2i\times i^2+3i=-i-2i+3i=0$ so $P(i)=0$

So according to the factor theorem, $(z-i)$ must be a factor of z^3+2iz^2+3i

b) $3^2-(5-i)\times 3+6-3i=9-15+3i+6-3i=0$

So as $P(3)=0$, then $(z-3)$ must be a factor of the polynomial

c) $2(i-2)^3+3(i-2)^2-(5+2i)(i-2)-17-9i$

$=2(i-2)(i^2-4i+4)+3(i^2-4i+4)-[5i-10-2-4i]-17-9i$

$=2(i-2)(3-4i)+3(3-4i)-[-12+i]-17-9i$

$=2[3i-6+4+8i]+9-12i+12-i-17-9i$

$=2[-2+11i]+4-22i=0$

d) $2(3-\sqrt{2}i)^4-12(3-\sqrt{2}i)^3+23(3-\sqrt{2}i)^2-6(3-\sqrt{2}i)+11$ (E)

$(3-\sqrt{2}i)^2=9-6\sqrt{2}i-2=7-6\sqrt{2}i$

$(3-\sqrt{2}i)^3=(3-\sqrt{2}i)(7-6\sqrt{2}i)=21-18\sqrt{2}i-7\sqrt{2}i-12=9-25\sqrt{2}i$

$(3-\sqrt{2}i)^4=(3-\sqrt{2}i)(9-25\sqrt{2}i)=27-75\sqrt{2}i-9\sqrt{2}i-50=-23-84\sqrt{2}i$

(E) $=2[-23-84\sqrt{2}i]-12[9-25\sqrt{2}i]+23[7-6\sqrt{2}i]-6[3-\sqrt{2}i]+11$

(E) $=\underbrace{-46-108+161-18+11}_{=0}+i\sqrt{2}\underbrace{[-168+300-138+6]}_{=0}$

2 Given $P(z) = z^3 - z^2 - z + a$, what is the value of a if $P\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$?

A 2 **B** 1 **C** -2 **D** -1

$-\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{2i\pi/3}$

(II quadrant)

so $z^3 - z^2 - z = e^{2i\pi} - e^{+i\frac{4\pi}{3}} - e^{+i2\pi/3}$

$\cancel{= 1 - \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}$

—— $= +1 - \left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] - \left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

—— $= 2$

So a must be equal to (-2) $\boxed{C}$

3 Given $P(z) = z^4 - 2z^3 + az - 9$, find the value of a if $P(1 + \sqrt{2}i) = 0$.

$z^4 - 2z^3 + az - 9 = 0$ for $z = 1 + \sqrt{2}i$

$(1+\sqrt{2}i)^2 = 1 + 2\sqrt{2}i - 2 = -1 + 2\sqrt{2}i$

$(1+\sqrt{2}i)^3 = (1+\sqrt{2}i)(-1+2\sqrt{2}i) = -1 + 2\sqrt{2}i - \sqrt{2}i - 4 = -5 + \sqrt{2}i$

$(1+\sqrt{2}i)^4 = (1+\sqrt{2}i)(-5+\sqrt{2}i) = -5 + \sqrt{2}i - 5\sqrt{2}i - 2 = -7 - 4\sqrt{2}i$

So $z^4 - 2z^3 + az - 9 = (-7 - 4\sqrt{2}i) - 2(-5 + \sqrt{2}i) + a(1 + \sqrt{2}i) - 9$

—— $= [-7 + 10 + a - 9] + i[-4\sqrt{2} - 2\sqrt{2}\cancel{i} + a\sqrt{2}]$

—— $= [-6 + a] + i\sqrt{2}[-6 + a]$

so $\boxed{a = 6}$

6 Factorise each polynomial over the set of complex numbers:

(a) $z^2 + 2z + 3$ (b) $2z^2 - 2z + 1$ (c) $2z^3 - 3z^2 + 2z - 3$

a) $\Delta = 4 - 4 \times 3 = -8 = [2\sqrt{2}i]^2$

So $z = \dfrac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i$

$z^2 + 2z + 3 = [z - (-1 + \sqrt{2}i)][z - (-1 - \sqrt{2}i)] = [z + 1 - \sqrt{2}i][z + 1 + \sqrt{2}i]$

b) $\Delta = 4 - 4 \times 2 = -4 = (2i)^2$

$z = \dfrac{2 \pm 2i}{2 \times 2} = \dfrac{1}{2} \pm \dfrac{i}{2}$

$2z^2 - 2z + 1 = 2\left(z^2 - z + \frac{1}{2}\right) = 2\left[z - \left(\frac{1}{2} + \frac{i}{2}\right)\right]\left[z - \left(\frac{1}{2} - \frac{i}{2}\right)\right]$

$= [2z - 1 - i][z - \frac{1}{2} + i/2]$

c) $2z^3 - 3z^2 + 2z - 3 = 2z^3 + 2z - 3z^2 - 3$

$= 2z(z^2 + 1) - 3(z^2 + 1)$

$= (z^2 + 1)(2z - 3)$

$= (2z - 3)(z - i)(z + i)$

8 Factorise $z^4 - 16$ over:

(a) the set of integers (b) the set of complex numbers.

a) $z^4 - 16 = (z^2-4)(z^2+4) = (z-2)(z+2)(z^2+4)$

b) $z^4 - 16 = (z-2)(z+2)(z-2i)(z+2i)$

10 Factorise $z^3 - 4z^2 + 9z - 10$ over:

$P(z) = z^3 - 4z^2 + 9z - 10$

(a) the set of real numbers (b) the set of complex numbers.

a) $P(1) = -4 \neq 0$ $P(-1) = -6 \neq 0$ $P(2) = 0$ YES!

So $z^3 - 4z^2 + 9z - 10 = (z-2)(z^2 - 2z + 5)$

b) $\Delta = 4 - 4 \times 5 = -16 = (4i)^2$

$z = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

So $P(z) = (z-2)[z-(1+2i)][z-(1-2i)]$

$P(z) = (z-2)[z-1-2i][z-1+2i]$

12 When factorised over the set of complex numbers, $z^4 + 2z^2 + 1$ becomes:

A $(z^2 - 1)^2$ NO

B $(z - i)^2(z + 1)^2$ NO

C $(z^2 + 1)^2$ NO unfinished But quite

D $(z - i)^2(z + i)^2$ YES

13 Factorise $z^6 - 1$ over:

(a) the set of real numbers

(b) the set of complex numbers.

a) 1 and -1 are real roots.

$z^6 - 1 = (z^3 - 1)(z^3 + 1) = (z-1)\underbrace{(z^2 + z + 1)}_{①}(z+1)\underbrace{(z^2 - z + 1)}_{②}$

b) for ① $\Delta = 1 - 4 = -3 = (\sqrt{3}i)^2$ $\quad z = \dfrac{-1 \pm \sqrt{3}i}{2}$

for ② $\Delta = 1 - 4 = -3 = (\sqrt{3}i)^2$ $\quad z = \dfrac{1 \pm \sqrt{3}i}{2}$

and factorise

14 Factorise $z^5 + 3z^4 - z - 3$ over:

(a) the set of real numbers

(b) the set of complex numbers.

a) $z^5 + 3z^4 - z - 3 = z^5 - z + 3z^4 - 3 = z(z^4-1) + 3(z^4-1)$

$= (z^4-1)(z+3) = (z+3)(z^2-1)(z^2+1)$

$= (z+3)(z-1)(z+1)(z^2+1)$

b) $= (z+3)(z-1)(z+1)(z-i)(z+i)$