USING VECTORS IN GEOMETRIC PROOFS

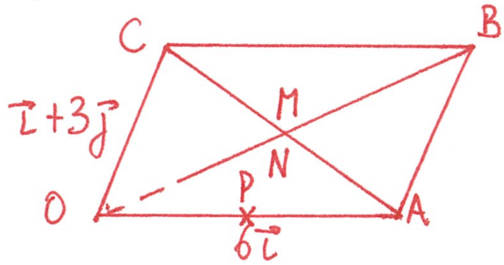
3 $OABC$ is a parallelogram in which $\overrightarrow{OA} = 6\underline{i}$ and $\overrightarrow{OC} = \underline{i} + 3\underline{j}$. Find:

(a) $\overrightarrow{AB}$ and $\overrightarrow{CB}$

(b) the diagonal vectors $\overrightarrow{OB}$ and $\overrightarrow{CA}$

(c) the vectors $\overrightarrow{ON}$ and $\overrightarrow{OM}$, where N is the midpoint of OB and M is the midpoint of CA. What conclusion can you make?

(d) the vectors $\overrightarrow{CP}$ and $\overrightarrow{BP}$, where P is the midpoint of OA.



a) $\overrightarrow{AB} = \overrightarrow{OC} = \vec{i} + 3\vec{j}$

$\overrightarrow{CB} = \overrightarrow{OA} = 6\vec{i}$

b) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 6\vec{i} + \vec{i} + 3\vec{j} = 7\vec{i} + 3\vec{j}$

$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA} = -\overrightarrow{OC} + \overrightarrow{OA} = -\vec{i} - 3\vec{j} + 6\vec{i} = 5\vec{i} - 3\vec{j}$

c) $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\left[7\vec{i} + 3\vec{j}\right] = \frac{7}{2}\vec{i} + \frac{3}{2}\vec{j}$

$\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \vec{i} + 3\vec{j} + \frac{1}{2}\overrightarrow{CA} = \vec{i} + 3\vec{j} + \frac{1}{2}(5\vec{i} - 3\vec{j})$

$\therefore \overrightarrow{OM} = \left(1 + \frac{5}{2}\right)\vec{i} + \left(3 - \frac{3}{2}\right)\vec{j} = \frac{7}{2}\vec{i} + \frac{3}{2}\vec{j}$

$\therefore \overrightarrow{ON} = \overrightarrow{OM}$ $\therefore$ N and M are identical.

d) $\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP} = -\overrightarrow{OC} + \frac{1}{2}\overrightarrow{OA}$

$\therefore \overrightarrow{CP} = -(\vec{i} + 3\vec{j}) + \frac{1}{2} \times 6\vec{i} = (3-1)\vec{i} - 3\vec{j} = 2\vec{i} - 3\vec{j}$

$\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP} = -\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OA} = -(7\vec{i} + 3\vec{j}) + \frac{1}{2} \times 6\vec{i}$

$\therefore \overrightarrow{BP} = (-7+3)\vec{i} - 3\vec{j} = -4\vec{i} - 3\vec{j}$

6 $OABC$ is a quadrilateral, $\overrightarrow{OA} = 4\underset{\sim}{i}$, $\overrightarrow{OB} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OC} = 8\underset{\sim}{j}$.

(a) If P and Q are the midpoints of AB and BC respectively, find $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$.

(b) Show that $\overrightarrow{PQ} = k\overrightarrow{AC}$. What geometrical conclusion can you now make?

a) P midpoint of AB, $\therefore$ $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = 4\vec{i} + \frac{1}{2}(-\overrightarrow{OA} + \overrightarrow{OB}) = 4\vec{i} + \frac{1}{2}(-4\vec{i} + 6\vec{i} + 2\vec{j})$

$\therefore \overrightarrow{OP} = 4\vec{i} + \frac{1}{2}(2\vec{i} + 2\vec{j}) = 5\vec{i} + \vec{j}$

Q midpoint of BC, $\therefore$ $\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{BO} + \overrightarrow{OC})$

$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = 6\vec{i} + 2\vec{j} + \frac{1}{2}(-(6\vec{i} + 2\vec{j}) + 8\vec{j})$

$\therefore \overrightarrow{OQ} = 6\vec{i} + 2\vec{j} + \frac{1}{2}(-6\vec{i} + 6\vec{j}) = 3\vec{i} + 5\vec{j}$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = -(5\vec{i} + \vec{j}) + (3\vec{i} + 5\vec{j})$

$\therefore \overrightarrow{PQ} = -2\vec{i} + 4\vec{j}$

b) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\overrightarrow{OA} + \overrightarrow{OC} = -(4\vec{i}) + 8\vec{j} = -4\vec{i} + 8\vec{j}$

$\therefore \overrightarrow{AC} = 2\,\overrightarrow{PQ}$ or $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{AC}$ are parallel, and with P and Q respectively midpoints of AB and BC, $|\overrightarrow{PQ}| = \frac{1}{2}|\overrightarrow{AC}|$

USING VECTORS IN GEOMETRIC PROOFS

7 The position vectors of the vertices A, B, C and D of a quadrilateral are $\underset{\sim}{i} + 2\underset{\sim}{j}$, $5\underset{\sim}{i} + 2\underset{\sim}{j}$, $4\underset{\sim}{i} - \underset{\sim}{j}$ and $2\underset{\sim}{i} - \underset{\sim}{j}$ respectively. Show that the diagonals intersect at right angles.

We need to show that $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\overrightarrow{OA} + \overrightarrow{OC}$

$\overrightarrow{AC} = -(\vec{i} + 2\vec{j}) + (4\vec{i} - \vec{j}) = 3\vec{i} - 3\vec{j}$

$\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD} = -\overrightarrow{OB} + \overrightarrow{OD}$

$\overrightarrow{BD} = -(5\vec{i} + 2\vec{j}) + (2\vec{i} - \vec{j}) = -3\vec{i} - 3\vec{j}$

$\overrightarrow{AC} \cdot \overrightarrow{BD} = (3\vec{i} - 3\vec{j}) \cdot (-3\vec{i} - 3\vec{j})$

$\quad = 3\vec{i} \cdot (-3\vec{i} - 3\vec{j}) - 3\vec{j} \cdot (-3\vec{i} - 3\vec{j})$

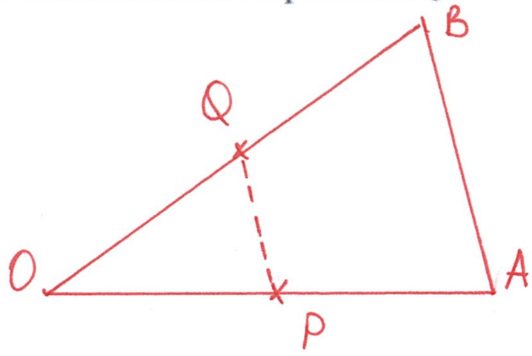
$\quad = -9 \qquad + 9$

$\quad = 0$

$\therefore$ the diagonals intersect at right angles.

USING VECTORS IN GEOMETRIC PROOFS

15 P and Q are the midpoints of the sides OA and OB of triangle OAB. Use a vector method to prove that PQ is parallel to AB and half its length.



$$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$$

$$\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OB}$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = -\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OB}$$

$$\overrightarrow{PQ} = -\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$$

$$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB}$$

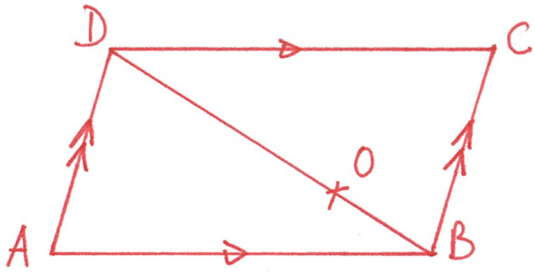
$$\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{AO} + \overrightarrow{OB}\right)$$

$$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$$

$\therefore$ PQ is parallel to AB and is half its length.

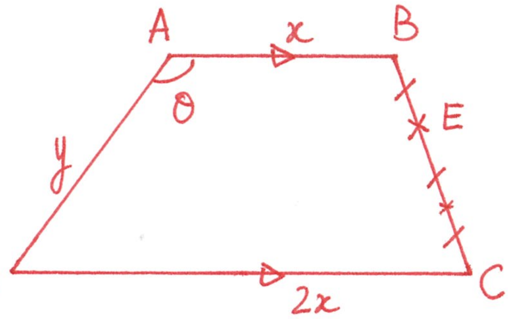
USING VECTORS IN GEOMETRIC PROOFS

17 O is any point on the diagonal BD of the parallelogram $ABCD$. Prove that $\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{CO} = \overrightarrow{DO}$



$$\begin{aligned}
\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{CO} &= (\overrightarrow{AO} + \overrightarrow{OB}) + \overrightarrow{CO} \\
&= \overrightarrow{AB} + \overrightarrow{CO} \\
&= \overrightarrow{DC} + \overrightarrow{CO} \\
&= \overrightarrow{DO}
\end{aligned}$$

21 $ABCD$ is a trapezium in which $AB = x$, $DC = 2x$, $DA = y$. If E is a point in BC such that $BE = \frac{1}{3}BC$, prove that $\overrightarrow{AC} \bullet \overrightarrow{DE} = \frac{2}{3}\left(4x^2 - y^2\right)$.



$$\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$$

$$\overrightarrow{AC} \cdot \overrightarrow{DE} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{DC} + \overrightarrow{CE}\right)$$

$$= \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(2\overrightarrow{AB} + \frac{2}{3}\overrightarrow{CB}\right)$$

$$= \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(2\overrightarrow{AB} - \frac{2}{3}\overrightarrow{BC}\right)$$

$$= 2\left|\overrightarrow{AB}\right|^2 - \frac{2}{3}\left|\overrightarrow{BC}\right|^2 + \overrightarrow{AB} \cdot \overrightarrow{BC}\left(2 - \frac{2}{3}\right)$$

$$= 2x^2 - \frac{2}{3}\left|\overrightarrow{BC}\right|^2 + \frac{4}{3}\overrightarrow{AB} \cdot \overrightarrow{BC}$$

* $\left|\overrightarrow{BC}\right|^2 = \overrightarrow{BC} \cdot \overrightarrow{BC}$. But $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{BA} + \overrightarrow{AD} + 2\overrightarrow{AB}$

$\therefore \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD}$

$\therefore \left|\overrightarrow{BC}\right|^2 = \left(\overrightarrow{AB} + \overrightarrow{AD}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{AD}\right) = x^2 + y^2 + 2xy\cos\theta$

* $\overrightarrow{AB} \cdot \overrightarrow{BC} = \overrightarrow{AB} \cdot \left(\overrightarrow{AB} + \overrightarrow{AD}\right) = x^2 + \overrightarrow{AB} \cdot \overrightarrow{AD} = x^2 + xy\cos\theta$

So $\overrightarrow{AC} \cdot \overrightarrow{DE} = 2x^2 - \frac{2}{3}\left(x^2 + y^2 + 2xy\cos\theta\right) + \frac{4}{3}\left(x^2 + xy\cos\theta\right)$

$$= \frac{8}{3}x^2 - \frac{2}{3}y^2$$

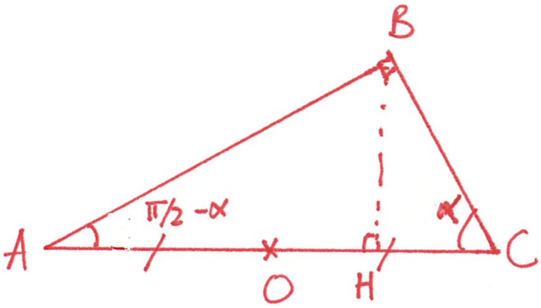
$$= \frac{2}{3}\left(4x^2 - y^2\right)$$

USING VECTORS IN GEOMETRIC PROOFS

Prove that:

26 The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices.

$$\vec{AO} = \frac{1}{2}\vec{AC}$$



We want to demonstrate that $|\vec{OB}| = |\vec{OA}| = |\vec{OC}|$

$$|\vec{OB}|^2 = \vec{OB}\cdot\vec{OB} = (\vec{OA}+\vec{AB})\cdot(\vec{OC}+\vec{CB})$$

$$= \vec{OA}\cdot\vec{OC} + \vec{OA}\cdot\vec{CB} + \vec{AB}\cdot\vec{OC} + \underbrace{\vec{AB}\cdot\vec{CB}}_{=0 \text{ as } \perp}$$

$$= -\vec{OA}\cdot\vec{CO} + |\vec{OA}||\vec{CB}|\cos\alpha + |\vec{AB}||\vec{OC}|\cos\left(\frac{\pi}{2}-\alpha\right)$$

$$= -|\vec{OA}|^2 + |\vec{OA}||\vec{CB}|\cos\alpha + |\vec{AB}||\vec{OA}|\cos\left(\frac{\pi}{2}-\alpha\right)$$

$$= -|\vec{OA}|^2 + |\vec{OA}|\left[|\vec{CB}|\cos\alpha + |\vec{AB}|\cos\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$= -|\vec{OA}|^2 + |\vec{OA}|\,[\underbrace{|\vec{HC}| + |\vec{AH}|}]$$

$$= -|\vec{OA}|^2 + |\vec{OA}|\quad|\vec{AC}|$$

$$= -|\vec{OA}|^2 + |\vec{OA}| \times 2|\vec{OA}|$$

$$= -|\vec{OA}|^2 + 2|\vec{OA}|^2$$

$$= |\vec{OA}|^2$$

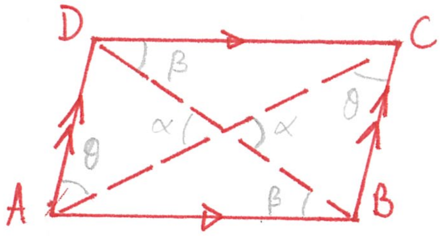
So $|\vec{OB}|^2 = |\vec{OA}|^2 \quad \therefore \quad |\vec{OB}| = |\vec{OA}|$

As $|\vec{OA}| = |\vec{OC}|$ then $|\vec{OB}| = |\vec{OA}| = |\vec{OC}|$

USING VECTORS IN GEOMETRIC PROOFS

Prove that:

27 If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.



Assume $|\overrightarrow{AC}| = |\overrightarrow{DB}|$

We want to show that $\overrightarrow{AD} \cdot \overrightarrow{AB} = 0$

let $\overrightarrow{AB} = \overrightarrow{DC} = \vec{a}$

and $\overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$

So $|\overrightarrow{AC}| = |\overrightarrow{DB}| \iff |\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{DA} + \overrightarrow{AB}|$

$\iff |\vec{a} + \vec{b}| = |-\vec{b} + \vec{a}|$

$\iff |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$\iff |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

$\iff (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

$\iff |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 = |\vec{a}|^2 - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + |\vec{b}|^2$

$\iff 2\vec{a} \cdot \vec{b} = -2\vec{a} \cdot \vec{b}$

$\iff 4\vec{a} \cdot \vec{b} = 0$

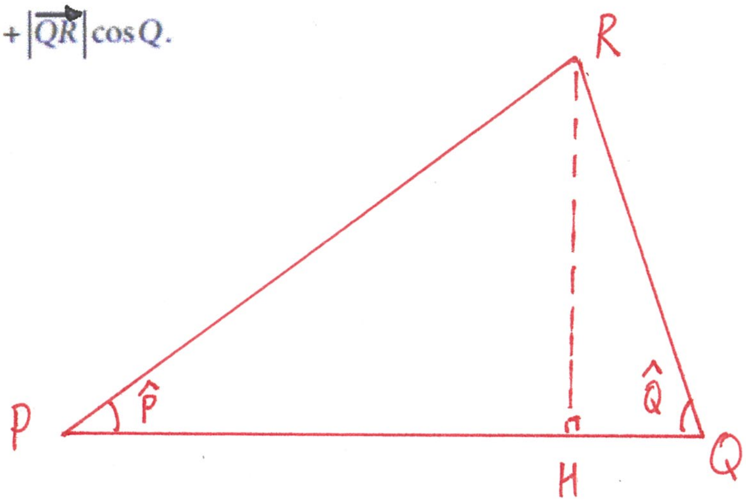
So $\vec{a} \cdot \vec{b} = 0$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular

The //gram is in fact a rectangle

Prove that:

31 For any triangle PQR, $|\overrightarrow{PQ}| = |\overrightarrow{PR}|\cos P + |\overrightarrow{QR}|\cos Q$.



$|\overrightarrow{PQ}| = |\overrightarrow{PH}| + |\overrightarrow{HQ}|$

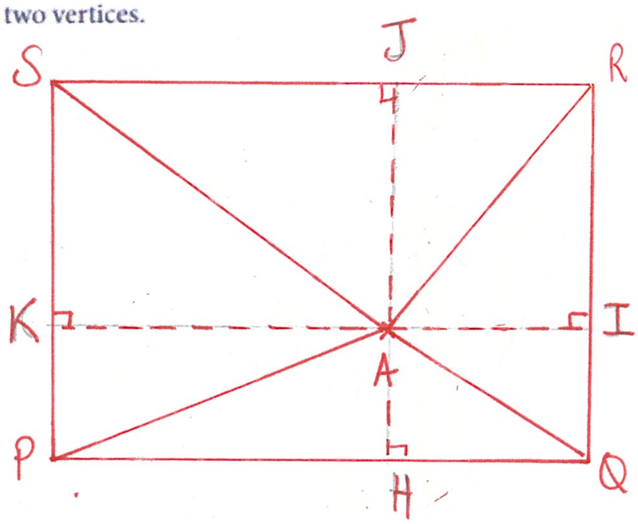
$= |\overrightarrow{PR}| \cos \hat{P} + |\overrightarrow{RQ}| \cos \hat{Q}$

$= |\overrightarrow{PR}| \cos P + |\overrightarrow{QR}| \cos Q$

USING VECTORS IN GEOMETRIC PROOFS

Prove that:

32 The sum of the squares of the distances from a point *A* to two opposite vertices of a rectangle is equal to the sum of the squares of the distances from *A* to the remaining two vertices.



We need to demonstrate that

$$|\vec{AS}|^2 + |\vec{AQ}|^2 = |\vec{AR}|^2 + |\vec{AP}|^2$$

$$|\vec{AS}|^2 = |\vec{SJ}|^2 + |\vec{AJ}|^2$$

$$|\vec{AQ}|^2 = |\vec{IQ}|^2 + |\vec{AI}|^2$$

So $|\vec{AS}|^2 + |\vec{AQ}|^2 = |\vec{SJ}|^2 + |\vec{AJ}|^2 + |\vec{IQ}|^2 + |\vec{AI}|^2$

Likewise $|\vec{AR}|^2 + |\vec{AP}|^2 = |\vec{AJ}|^2 + |\vec{JR}|^2 + |\vec{PH}|^2 + |\vec{AH}|^2$

But $|\vec{JR}| = |\vec{AI}|$, $|\vec{PH}| = |\vec{SJ}|$, $|\vec{AH}| = |\vec{IQ}|$

So $|\vec{AR}|^2 + |\vec{AP}|^2 = |\vec{AJ}|^2 + |\vec{AI}|^2 + |\vec{SJ}|^2 + |\vec{IQ}|^2$

$$\therefore \quad |\vec{AS}|^2 + |\vec{AQ}|^2 = |\vec{AR}|^2 + |\vec{AP}|^2$$