MATHEMATICAL INDUCTION INVOLVING SERIES

Mathematical induction is a method to prove a statement containing integers, for example the statement:

$$1+2+3+\cdots+(n-1)+n=\frac{n(n+1)}{2}$$

A proof by induction consists of a **three-step process**, the final step being a conclusion:

Step 1: Prove that the statement is true for the smallest possible value of n

in the example above, for $n=2$, we have indeed $1+2=\frac{2(2+1)}{2}$

Step 2: Prove that if the statement is true for $n=k$, then it must also be true for the next value $n=k+1$

in the example above, we assume that the statement is true for $n=k$, i.e.:

$$1+2+3+\cdots+(k-1)+k=\frac{k(k+1)}{2}$$

In that case:

$$1+2+3+\cdots+(k-1)+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$1+2+3+\cdots+(k-1)+k+(k+1)=\frac{k(k+1)+2(k+1)}{2}$$

$$1+2+3+\cdots+(k-1)+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

which is the same formula than the original statement.

Step 3: Write a conclusion, such as:

- The statement is true for $n=k+1$ if it is true for $n=k$
- The statement is true for the smallest possible value of n
- Therefore the statement is true for all possible values of n

For the example above, we would say:

- The statement $1+2+3+\cdots+(n-1)+n=\frac{n(n+1)}{2}$ is true for $n=k+1$ if it is true for $n=k$
- The statement is true for the smallest possible value of $n=2$
- Therefore the statement is true for all possible values of n

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Example 1

Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all integers $n \geq 1$.

Solution

Step 1 Prove that the statement is true for $n = 1$.

When $n = 1$: LHS $= 1$ RHS $= 1^2 = 1$

LHS = RHS $\therefore$ the statement is true for $n = 1$

Step 2 Assume the statement is true for $n = k$, where k is any integer greater than or equal to 1.

i.e. assume that $\boxed{1 + 3 + 5 + \ldots + (2k - 1)} = k^2$ [a]

Now prove that the statement will be true for $n = k + 1$ if it is true for $n = k$.

i.e. prove that $1 + 3 + 5 + \ldots + (2[k + 1] - 1) = (k + 1)^2$

$$\text{LHS} = 1 + 3 + 5 + \ldots + (2[k + 1] - 1)$$

$$= \boxed{1 + 3 + 5 + \ldots + (2k - 1)} + (2[k + 1] - 1)$$

using [a]: $= k^2 + (2k + 1)$

(You can only prove this is true when $n = k + 1$ if it is true when $n = k$.)

$$= (k + 1)^2$$

$$= \text{RHS}$$

Step 3 Conclusion

The statement is true for $n = k + 1$ if it is true for $n = k$ (Step 2)

The statement is true for $n = 1$ (Step 1)

$\therefore$ by induction, the statement is true for all integers $n \geq 1$.

Example 2

Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Solution

Let $S(n)$ be the statement that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$, where n is a positive integer.

Step 1 Prove that $S(1)$ is true.

$$\text{LHS} = 1^2 = 1 \qquad \text{RHS} = \frac{1(1+1)(2\times 1+1)}{6} = \frac{1(2)(3)}{6} = 1$$

$$\text{LHS} = \text{RHS} \qquad \therefore S(1) \text{ is true}$$

Step 2 Assume $S(k)$ is true for a positive integer k.

i.e. assume that $\boxed{1^2 + 2^2 + 3^2 + \ldots + k^2} = \frac{k(k+1)(2k+1)}{6}$ [a]

Now, prove that $S(k + 1)$ is true if $S(k)$ is true.

i.e. prove that $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\text{LHS} = \boxed{1^2 + 2^2 + 3^2 + \ldots + k^2} + (k+1)^2$$

using [a]: $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

Now look at the RHS: you want $\frac{(k+1)}{6}$ as a factor.

$$\text{LHS} = \frac{(k+1)}{6}(k(2k+1)+6(k+1))$$

$$= \frac{(k+1)}{6}\left(2k^2+7k+6\right)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}$$

Step 3 Conclusion

$S(k + 1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 3

Prove by induction that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for all integers $n \geq 1$.

Solution

Let $S(n)$ be the statement that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for positive integer n.

Step 1 Prove that $S(1)$ is true.

$$\text{LHS} = 1 + (2 \times 1) \quad \text{(Note that the sum on the LHS starts with } n \text{ and finishes with } 2n.\text{)}$$
$$= 3$$

$$\text{RHS} = \frac{3 \times 1(1+1)}{2} = 3$$

LHS = RHS $\quad \therefore S(1)$ is true

Step 2 Assume $S(k)$ is true for an integer $k \geq 1$.

i.e. assume that $\boxed{k + (k+1) + (k+2) + \ldots + 2k} = \frac{3k(k+1)}{2}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $(k+1) + (k+2) + \ldots + 2(k+1) = \frac{3(k+1)((k+1)+1)}{2}$

Note that when $n = k+1$, the sum on the LHS starts with $(k+1)$ and finishes with $2(k+1)$.

$$\text{LHS} = (k+1) + (k+2) + \ldots + 2k + (2k+1) + (2k+2)$$

Now you have a problem! At this point you need to use the substitution of line [a], but you are missing the first term (the k). In going from $S(k)$ to $S(k+1)$ you have lost the first term but gained two extra terms on the LHS.The solution is to break up the extra two terms $(2k+1) + (2k+2)$ as a k term and a $(3k+3)$ term.

$$\text{LHS} = \boxed{k + (k+1) + (k+2) + \ldots + 2k} + (3k+3)$$

using [a]: $= \frac{3k(k+1)}{2} + 3(k+1)$

$= \frac{3(k+1)}{2}[k+2]$ taking the common factor of $\frac{3(k+1)}{2}$

= RHS

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.