COMPOUND INTEREST APPLICATIONS

1 At the beginning of 2013, Amélie invests $1000. If compound interest is paid at 6% p.a., the value of the investment after 8 years is:

A $60 **B** $480 $\boxed{\text{C}}$ $1593.85 **D** $10 491.32

$$A_n = P(1+r)^n = 1{,}000 \times (1+0.06)^8 = 1593.85$$

$\boxed{4}$ 2 Write a recursion relation for each of the following.

(a) An investment of $1000 at an interest of 3.1% with the interest compounded annually.
(b) A loan of $5000 at an interest rate of 6.2% with the interest compounded six-monthly.
(c) An investment of $10 000 at an interest of 2.7% with the interest monthly.

a) $A_n = 1{,}000 \times (1+0.031)^n$ or $A_n = A_{n-1} \times 1.031$

b) $A_n = 5{,}000 \times \left(1+\frac{0.062}{2}\right)^n$ where n is the number of periods

or $A_n = A_{n-1} \times \left(1+\frac{0.062}{2}\right)$

c) $A_n = 10{,}000 \times \left(1+\frac{0.027}{12}\right)^n$ where n is the number of periods.

or $A_n = A_{n-1} \times \left(1+\frac{0.027}{12}\right)$

COMPOUND INTEREST APPLICATIONS

3 What is the annual compound interest rate for the information given in each recurrence relation?

(a) $A_0 = 5000$, $A_n = 1.027 \times A_{n-1}$ if the interest is charged annually.
(b) $A_0 = 10\,000$, $A_n = 1.015 \times A_{n-1}$ if the interest is charged six-monthly.
(c) $A_0 = 4000$, $A_n = 1.005 \times A_{n-1}$ if the interest is charged monthly.

a) interest rate is 2.7% p.a

b) interest rate is 1.5% per six-months
so 3% p.a

c) interest rate is 0.5% per month
so $0.5 \times 12 = 6\%$ p.a

4 In 2006, 5000 students entered for a particular examination. The number increased each year by 20% of the number who entered the previous year. Calculate:

(a) the number who entered in 2011
(b) the total number who entered between 2006 and 2011 inclusive.

a) $N_n = 5000 \times 1.2^n$ where n is the number of years after 2006

So $N_{(2011-2006)} = N_5 = 5000 \times 1.2^5 = 12{,}442$ to the nearest integer.

b) $S = N_0 + N_1 + N_2 + N_3 + N_4 + N_5 = N_0 + N_0R + N_0R^2 + N_0R^3 + N_0R^4 + N_0R^5$

$= N_0(1 + R + R^2 + R^3 + R^4 + R^5)$

$= N_0\left[\dfrac{R^6 - 1}{R - 1}\right]$

So $S = 5000 \times \left[\dfrac{1.2^6 - 1}{1.2 - 1}\right] = 49{,}650$

6 At the beginning of 2011, a mining town had a population of 15000. It was estimated that this would increase each year by 8% of its population at the beginning of the year. What should the population be at the beginning of 2019?

$P_n = 15{,}000 \times (1 + 0.08)^n$ n is the number of years after 2011.

$P_{(2019-2011)} = 15{,}000 \times (1+0.08)^{(2019-2011)}$

$P_{(2019-2011)} = 15{,}000 \times 1.08^8 = 27{,}764$ to the nearest integer.

7 The value of a new car is \$35000. Its value depreciates (falls) each year by 15% of its value at the beginning of that year. After how many years will its value be \$15000?

$V_n = 35{,}000 \times (1-0.15)^n$

So $15{,}000 = 35{,}000 \times 0.85^n$

$\therefore 0.85^n = \frac{15}{35} = \frac{3}{7}$

$\therefore n = \frac{\ln(3/7)}{\ln(0.85)} = 5.2$

between 5 and 6 years.

COMPOUND INTEREST APPLICATIONS

9 Kris contributes towards a pension for his retirement by depositing into a superannuation fund an amount of \$5000 on each of his 44 birthdays from his 21st to his 64th inclusive.

(a) If the money is invested at 7% p.a. compound interest, how much money is in Kris's superannuation fund on his 65th birthday?

(b) As Kris is about to make the deposit on his 43rd birthday, he is told that he can only expect to earn 3% p.a. compound interest for the remainder of the time. How much less will be in the superannuation fund on his 65th birthday?

a) $S = 5{,}000 \times 1.07 + 5{,}000 \times 1.07^2 + 5{,}000 \times 1.07^3 + \dots + 5{,}000 \times 1.07^{44}$

$S = 5{,}000 \times 1.07 \times [1 + 1.07 + 1.07^2 + \dots + 1.07^{43}]$

$S = 5{,}000 \times 1.07 \left[\frac{1.07^{44}-1}{1.07-1}\right] = 1{,}423{,}747$

b) He has made payments from his 21st birthday to his 42nd birthday, so 22 payments. Each of those 22 payments gets them 3% interest over the next 22 birthdays.

$S = 5{,}000 \times 1.07^{22} \times 1.03^{22} + 5{,}000 \times 1.07^{21} \times 1.03^{22} + 5{,}000 \times 1.07^{20} \times 1.03^{22} +$

$\dots 5{,}000 \times 1.07 \times 1.03^{22} + 5{,}000 \times 1.03^{22} + 5{,}000 \times 1.03^{21} + \dots + 5{,}000 \times 1.03$

$S = 5{,}000 \times 1.03^{22} [1 + 1.07 + \dots + 1.07^{22}] + 5{,}000 \times 1.03 \times [1 + \dots + 1.03^{20}]$

$S = 5{,}000 \times 1.03^{22} \times \left[\frac{1.07^{23}-1}{1.07-1}\right] + 5{,}000 \times 1.03 \times \left[\frac{1.03^{21}-1}{1.03-1}\right]$

$S = 511{,}946 + 147{,}684 = 659{,}630$

which is $(1{,}423{,}747 - 659{,}630) = 764{,}117$ less than what he had at a)

COMPOUND INTEREST APPLICATIONS

12 Kathryn borrows \$200 000, to be repaid in equal monthly instalments. The interest rate is 8.4% p.a., calculated monthly.

(a) Show that the interest for the first month is \$1400.
(b) Why should Kathryn's payments be more than \$1400 per month?
(c) Kathryn decides to pay \$3000 per month off the loan. Show that the amount owing after two repayments, $\$A_2$, is given by $A_2 = 200\,000 \times 1.007^2 - 3000(1 + 1.007)$.
(d) Hence find an expression for $\$A_n$, the amount owing after the nth repayment.
(e) How long will it take Kathryn to pay off the loan?

a) $I = 200{,}000 \times \frac{0.084}{12} = 1{,}400$

b) because 1,400 is the interest only. If she only repays \$1,400 per month, she doesn't reimburse any of the capital.

c) After 1 month, she owes $200{,}000 + 1{,}400 - 3{,}000 = 198{,}400$

After 2 months, she owes $198{,}400 + 198{,}400 \times \frac{0.084}{12} - 3000 = 196788.80$

i.e After 1 month, she owes $200{,}000 + 200{,}000 \times \frac{0.084}{12} - 3000 = A_1$

after 2 months, she owes $A_1 + A_1 \times \frac{0.084}{12} - 3000 = A_2$

So $A_2 = A_1\left[1 + \frac{0.084}{12}\right] - 3000 = \left[200{,}000\left[1 + \frac{0.084}{12}\right] - 3000\right]\left[1 + \frac{0.084}{12}\right] - 3000$

$A_2 = \left[200{,}000 \times 1.007 - 3{,}000\right](1.007) - 3{,}000$

$A_2 = 200{,}000 \times 1.007^2 - 3000(1 + 1.007)$

d) $A_n = 200{,}000 \times 1.007^n - 3{,}000(1 + 1.007 + \ldots + 1.007^{n-1})$

$A_n = 200{,}000 \times 1.007^n - 3{,}000\left[\frac{1.007^n - 1}{0.007}\right]$

e) $A_n = 0$ when $200{,}000 \times 1.007^n = 3{,}000\left[\frac{1.007^n - 1}{0.007}\right]$

$\Leftrightarrow \frac{1400}{3{,}000} \times 1.007^n - 1.007^n = -1$

$\Leftrightarrow 1.007^n\left[1 - \frac{1400}{3000}\right] = 1 \Leftrightarrow 1.007^n = \frac{15}{8}$

so $n = \frac{\ln(15/8)}{\ln(1.007)} = 90$ years.

COMPOUND INTEREST APPLICATIONS

13 Phuong decides to set up a trust fund to provide for the education of her grandchildren. She deposits \$60 at the beginning of each month into an account that earns 7.2% p.a. compounded monthly. The trust fund will mature at the end of the month of her final investment, 20 years after her first investment, so Phuong will need to make 240 monthly deposits.

(a) After 20 years, what is the value of the first \$60 deposited?

(b) Write a geometric series for the value of all the deposits and calculate the final value of the trust fund.

a) $P = 60\left(1 + \frac{7.2}{100 \times 12}\right)^{240} = 252.15$

b) $\frac{7.2}{100 \times 1.2} = 0.006$ so

$P = 60 \times 1.006^{240} + 60 \times 1.006^{239} + 60 \times 1.006^{238} + \dots + 60 \times 1.006$

$P = 60 \times 1.006 \left[1 + 1.006 + \dots + 1.006^{239}\right]$

$P = 60 \times 1.006 \left[\frac{1.006^{240} - 1}{1.006 - 1}\right]$

$P = 32{,}217.89$