PARAMETRIC FORM OF A FUNCTION OR RELATION

For questions **1** to **14**, find the Cartesian equation of the curves with the parametric equations given.

1 $x = 2t, y = t + 2$ **2** $x = t, y = t^2$ **3** $x = t, y = \frac{1}{t}$ **4** $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le 2\pi$

① $t = x/2$ $y = \frac{x}{2} + 2$ line

② $y = x^2$ parabola

③ $y = \frac{1}{x}$ hyperbola

④ $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{2}$

so $\cos^2\theta = \left(\frac{x}{2}\right)^2$ and $\sin^2\theta = \left(\frac{y}{2}\right)^2$ so $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$\Leftrightarrow x^2 + y^2 = 4$ Circle centred on origin, of radius 2

5 $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le \pi$ **6** $x = t + 3, y = t^2 - 5, t \ge 0$ **7** $x = 2u - 2, y = 3u + 1, 1 \le u \le 3$

⑤ $\frac{x^2}{4} = \cos^2\theta$, $\frac{y^2}{4} = \sin^2\theta$ so $x^2 + y^2 = 4$ but $0 \leq \theta \leq \pi$ so half circle.

⑥ $t = x - 3$ so $y = (x-3)^2 - 5 = x^2 - 6x + 4$
with $x \geq 3$ only part of a parabola.

⑦ $x = 2u - 2$ so $x + 2 = 2u$ so $u = \frac{x+2}{2}$

$y = 3\left(\frac{x+2}{2}\right) + 1 = \frac{3}{2}x + 4$ with $1 \leq u \leq 3$ so $1 \leq \frac{x+2}{2} \leq 3$

$\Leftrightarrow 2 \leq x + 2 \leq 6 \Leftrightarrow 0 \leq x \leq 4$

part of a line.

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8 $x = v^3, y = 1 - v^2, -1 \le v \le 1$ **9** $x = t + 2, y = t^2 - 1$ **10** $x = \cos t, y = \cos t, 0 \le t \le 2\pi$ **11** $x = 2t^2, y = 4t$

(8) $v = x^{1/3}$ so $y = 1 - x^{2/3}$ with $-1 \leqslant v \leqslant 1$ so $-1 \leqslant x^{1/3} \leqslant 1$

$\therefore -1 \leqslant x \leqslant 1$

(9) $t = x - 2$ so $y = (x-2)^2 - 1 = x^2 - 4x + 3$ parabola.

(10)

$x^2 + y^2 = 1$ circle centre $(0,0)$, radius 1

(11) $y = 4t$ so $t = y/4$

$x = 2t^2 = 2\left(\frac{y}{4}\right)^2 = \frac{2y^2}{16} = \frac{y^2}{8}$

so $y^2 = 8x$ with $x \geqslant 0$

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12 $x = 2\cos\theta, y = \sqrt{3}\sin\theta, 0 \le \theta \le 2\pi$ **13** $x = 2\cos t, y = \sin t, 0 \le t \le \pi$ **14** $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

(12) $\cos\theta = \frac{x}{2}$ so $\cos^2\theta = \left(\frac{x}{2}\right)^2$ $y = \sqrt{3}\sin\theta$ so $\sin\theta = \frac{y}{\sqrt{3}}$

$\sin^2\theta = \frac{y^2}{3}$

So $\left(\frac{x}{2}\right)^2 + \frac{y^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 = 12$

(13) $\cos t = \frac{x}{2}$ $\sin t = y$ so $\left(\frac{x}{2}\right)^2 + y^2 = 1$

or $x^2 + 4y^2 = 4$ with $y \geq 0$ $y = \sqrt{1 - \frac{x^2}{4}}$

(14) if $t = \tan\theta/2$ $x = \frac{2\tan\theta/2}{1+\tan^2\theta/2} = \frac{\frac{2\sin\theta/2}{\cos\theta/2}}{\sec^2\theta/2} = 2\sin(\theta/2)\cos(\theta/2) = \sin\theta.$

and $y = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)} = \frac{1 - \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}}{\sec^2(\theta/2)} = \frac{\frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2)}}{\frac{1}{\cos^2(\theta/2)}}$

$y = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos\left(2 \times \frac{\theta}{2}\right) = \cos\theta.$

So $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$

Circle centre $(0,0)$ radius 1.

15 Two boats on a lake start sailing at the same time. Boat A moves on a course given by $x = \frac{t}{2}, y = t + 1$, while boat B moves on a course given by $x = t - 2, y = -2t + 9$, where t is the time elapsed in hours.

(a) Find the Cartesian equation for the course of each boat. Show that the courses intersect at the point $(1, 3)$.

(b) Do the boats collide? Justify your answer.

$\begin{cases} x_A = t/2 \\ y_A = t+1 \end{cases}$ $\begin{cases} x_B = t-2 \quad t = x_B + 2 \\ y_B = -2t+9 \end{cases}$

a) $y_A = 2x_A + 1$

$y_B = -2(x_B+2)+9 = -2x_B + 5$

when $x_A = 1$, $y_A = 2\times1+1 = 3$ whereas when $x_B = 1$, $y_B = -2\times1+5 = 3$

So the courses intersect.

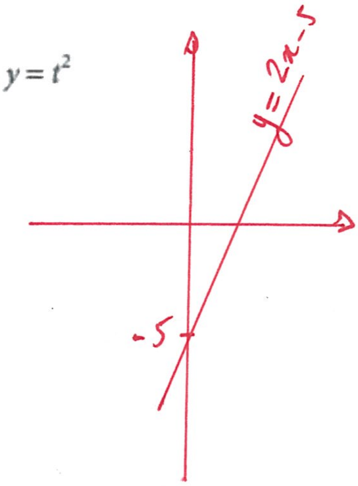
b) when $x_A = 1$, $t = 2 \times x_A = 2$ whereas when $x_B = 1$, $t = 1+2 = 3$

So they don't collide.

18 Sketch the graph of each curve from its parametric equations.

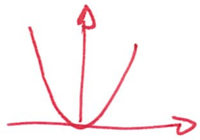
(a) $x = t + 1, y = 2t - 1$ (b) $x = t, y = 2t^2$ (c) $x = \frac{t}{2}, y = t^2$

(d) $x = 4\sin\theta, y = 4\cos\theta$ (e) $x = \frac{\sin\theta}{2}, y = \frac{\cos\theta}{2}$



a) $t = x-1$ so $y = 2(x-1)-1 = 2x-5$

b) $y = 2x^2$ parabola



c) $y = (2x)^2 = 4x^2$ parabola.

d)

$\sin\theta = \frac{x}{4}$ and $\cos\theta = \frac{y}{4}$

so $\sin^2\theta = \left(\frac{x}{4}\right)^2$ and $\cos^2\theta = \left(\frac{y}{4}\right)^2$

so $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \Rightarrow x^2+y^2 = 4 = 2^2$

circle centre O, radius 2

e) $\sin\theta = 2x$ $\cos\theta = 2y$

so as $\sin^2\theta + \cos^2\theta = 1$, then

$(2x)^2 + (2y)^2 = 1 \Leftrightarrow x^2+y^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

Circle centred on O, of radius $\left(\frac{1}{2}\right)$