DERIVATIVES OF LOGARITHMIC FUNCTIONS

1 Differentiate:

(a) $\log_e 2x$ (b) $2\log_e x$ (c) $\log_e x^2$

(d) $\log_e(3x-5)$ (e) $\log_e x+3$ (f) $x^2-\log_e(4x-1)$

a) $f(x) = \ln 2x = g(h(x))$ with $g(x) = \ln x$ and $h(x) = 2x$

$g'(x) = \frac{1}{x}$ $h'(x) = 2$

So with the Chain rule

$f'(x) = g'[h(x)] \times h'(x) = \frac{1}{2x} \times 2 = \frac{1}{x}$

b) $f(x) = 2\ln x$ $f'(x) = \frac{2}{x}$

c) $f(x) = \ln x^2 = 2\ln x$ so $f'(x) = 2/x$

d) $f(x) = \ln(3x-5) = g[h(x)]$ with $g(x) = \ln x$ and $h(x) = 3x-5$

$g'(x) = \frac{1}{x}$ $h'(x) = 3$

$f'(x) = \frac{1}{3x-5} \times 3 = \frac{3}{3x-5}$

e) $f(x) = \ln x + 3$ $f'(x) = \frac{1}{x}$

f) $f(x) = x^2 - \ln(4x-1)$

$f'(x) = 2x - \frac{1}{4x-1} \times 4$

$f'(x) = 2x - \frac{4}{4x-1}$

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3 The derivative of $\log_e(3x^2+1)$ is:

$f(x) = \ln(3x^2+1)$

A $6x$ **B** $\frac{6}{x}$ **C** $\frac{6x}{3x^2+1}$ **D** $\frac{1}{x^3+x}$

$f'(x) = \frac{1}{3x^2+1} \times \cancel{6}x$ so $\boxed{C}$

4 Differentiate:

(a) $x\ln x$ **(b)** $x^3\ln x$ **(c)** $(x+2)\ln(x+2)$ **(d)** $(x^2+1)\ln 2x$

Product rule

a) $f(x) = x\ln x = u(x) \times v(x)$ $\quad f'(x) = \frac{x}{x} + 1 \times \ln x = 1 + \ln x$

b) $f(x) = x^3 \ln x = u(x) \times v(x)$

$u(x) = x^3$ $\quad u'(x) = 3x^2$ $\quad v(x) = \ln x$ $\quad v'(x) = 1/x$

So $f'(x) = 3x^2 \ln x + x^3 \times \frac{1}{x} = x^2[3\ln x + 1]$

c) $f(x) = (x+2)\ln(x+2)$

$f'(x) = (x+2) \times \frac{1}{x+2} + 1 \times \ln(x+2) = \ln(x+2) + 1$

d) $f(x) = (x^2+1)\ln 2x$

$f'(x) = (x^2+1) \times \frac{1 \times 2}{2x} + 2x \times \ln 2x$

$f'(x) = \frac{x^2+1}{x} + 2x \ln 2x$

$f'(x) = 2x\ln x + x + \frac{1}{x}$

DERIVATIVES OF LOGARITHMIC FUNCTIONS

(i) $\frac{\log_e x}{x}$ (j) $\frac{\log_e x}{e^x}$ (k) $\frac{\log_e(x^2+1)}{x}$ (l) $e^x \log_e(e^x+1)$

i) $f(x) = \frac{\ln x}{x} = \frac{u(x)}{v(x)}$ $\quad u(x) = \ln x \quad u'(x) = 1/x$

$v(x) = x \quad v'(x) = 1$

$$f'(x) = \frac{\frac{1}{x} \times x - 1 \times \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

j) $f(x) = \frac{\ln x}{e^x}$ $\quad u(x) = \ln x \quad u'(x) = 1/x$

$v(x) = e^x \quad v'(x) = e^x$

$$f'(x) = \frac{\frac{1}{x} \times e^x - e^x \ln x}{(e^x)^2} = \frac{\frac{1}{x} - \ln x}{e^x}$$

k) $f(x) = \frac{\ln(x^2+1)}{x}$ $\quad u(x) = \ln(x^2+1) \quad u'(x) = \frac{1}{x^2+1} \times 2x$

$v(x) = x \quad v'(x) = 1$

$$f'(x) = \frac{\frac{2x}{x^2+1} \times x - \ln(x^2+1)}{x^2}$$

$$f'(x) = \frac{2}{x^2+1} - \frac{\ln(x^2+1)}{x^2}$$

l) $f(x) = e^x \ln(e^x+1)$ $\quad u(x) = e^x \quad u'(x) = e^x$

$v(x) = \ln(e^x+1) \quad v'(x) = \frac{1}{e^x+1} \times e^x$

So $f'(x) = e^x \ln(e^x+1) + \frac{(e^x)^2}{(e^x+1)}$

$$f'(x) = e^x \left[\ln(e^x+1) + \frac{e^x}{e^x+1} \right]$$

DERIVATIVES OF LOGARITHMIC FUNCTIONS

7 If $f(x) = \log_e x$, find: **(a)** $f'(x)$ **(b)** $f''(x)$ **(c)** $f'(2)$ **(d)** $f''(2)$

a) $f'(x) = 1/x$ b) $f'(x) = x^{-1}$ so $f''(x) = (-1) \times x^{-2} = \frac{-1}{x^2}$

c) $f'(2) = \frac{1}{2}$

d) $f''(2) = -\frac{1}{4}$

8 Find the equation of the tangent and normal to the curve $y = \log_e x$ at the point where it crosses the x-axis.

$\frac{dy}{dx} = \frac{1}{x}$ $y = \ln x$ crosses the x axis when $y = 0$, i.e. $x = 1$

So $\frac{dy}{dx}$ is equal to $\frac{1}{1}$ at $x = 1$

$y - 0 = 1(x - 1)$ so $\boxed{y = x - 1}$

12 Solve: **(a)** $e^x = 2$ **(b)** $e^{3x} = 5$ **(c)** $e^{2x+3} = 7$ **(d)** $e^{x^2-1} = 10$

a) We take the neperian log on both sides: $\ln e^x = \ln 2 \therefore x = \ln 2$

b) $\ln e^{3x} = \ln 5$ so $3x = \ln 5$ $x = \frac{\ln 5}{3}$

c) $\ln e^{2x+3} = \ln 7$ so $(2x + 3) = \ln 7$ $2x = \ln 7 - 3$

$x = \frac{\ln 7 - 3}{2}$

d) $\ln e^{x^2-1} = \ln(10)$ so $x^2 - 1 = \ln 10$ $x^2 = 1 + \ln 10$

$x = \pm\sqrt{1 + \ln 10}$

13 Differentiate:

(a) $y = \log_e\left(\frac{x^3-1}{x}\right)$ **(b)** $f(x) = \log_e\left(e^x(x+2)\right)$ **(c)** $y = \log_e\left(\sqrt{x}(x+1)^5\right)$

a) $f(x) = \ln\left[\frac{x^3-1}{x}\right] = \ln(x^3-1) - \ln x$

$f'(x) = \frac{1}{x^3-1} \times 3x^2 - \frac{1}{x} = \frac{3x^2}{x^3-1} - \frac{1}{x}$

b) $f(x) = \ln\left[e^x(x+2)\right] = \ln e^x + \ln(x+2) = x + \ln(x+2)$

$f'(x) = 1 + \frac{1}{x+2} = \frac{x+2 \quad + 1}{x+2} = \frac{x+3}{x+2}$

c) $f(x) = \ln\left[\sqrt{x}(x+1)^5\right] = \ln\left[\sqrt{x}\right] + \ln\left[(x+1)^5\right]$

$f(x) = \ln x^{1/2} + 5\ln(x+1)$

$f(x) = \frac{1}{2}\ln x + 5\ln(x+1)$

So $f'(x) = \frac{1}{2} \times \frac{1}{x} + 5 \times \frac{1}{x+1} \times 1$

$f'(x) = \frac{1}{2x} + \frac{5}{x+1}$

$f'(x) = \frac{(x+1) + 5 \times 2x}{2x(x+1)}$

$f'(x) = \frac{11x+1}{2x(x+1)}$

15 Differentiate:

(a) a^{-x} **(b)** $a^x \log_a x$ **(c)** $\dfrac{\log_a x}{a^x}$ **(d)** $\sqrt{\log_a x}$

a) We know that $a^x = e^{x\ln a}$ so $f(x) = e^{-x\ln a} = (e^x)^{-\ln a}$

$f(x) = g[h(x)]$ with $g(x) = X^{-\ln a}$ and $h(x) = e^x$

$g'(x) = (-\ln a)X^{-\ln a - 1}$ $h'(x) = e^x$

$f'(x) = -\ln a\,(e^x)^{-(\ln a+1)} \times e^x = -\ln a\; e^{-x\ln a - x + x}$

$f'(x) = -\ln a\; e^{-x\ln a} = -\ln a \times a^{-x} = -a^{-x} \times \ln a$

OR more simply, as $(a^x)' = a^x \ln a$ so $(a^{-x})' = -a^{-x}\ln a$

b) $f(x) = a^x \times \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a}\left[a^x \ln x\right]$

$f'(x) = \dfrac{1}{\ln a}\left[a^x \ln a \times \ln x + a^x \times \dfrac{1}{x}\right] = a^x\left[\ln x + \dfrac{1}{x\ln a}\right]$

c) $f(x) = \dfrac{\log_a x}{a^x} = \dfrac{u(x)}{v(x)}$ $u(x) = \log_a x$ $u'(x) = 1/x\ln a$

$v(x) = a^x$ $v'(x) = a^x \ln a$

So $f'(x) = \dfrac{\dfrac{a^x}{x\ln a} - a^x \ln a \times \log_a x}{(a^x)^2} = \dfrac{\dfrac{1}{x\ln a} - \ln a \times \dfrac{\ln x}{\ln a}}{a^x}$

$f'(x) = \dfrac{\dfrac{1}{x\ln a} - \ln x}{a^x} = \dfrac{1 - x\ln a \ln x}{x a^x \ln a}$

d) $f(x) = (\log_a x)^{1/2} = \left(\dfrac{\ln x}{\ln a}\right)^{1/2} = (\ln x)^{1/2} \times \dfrac{1}{(\ln a)^{1/2}}$

$g(x) = x^{1/2}$ $g'(x) = \dfrac{1}{2}x^{-1/2}$ $h(x) = \ln x$ $h'(x) = 1/x$

$f'(x) = \dfrac{1}{(\ln a)^{1/2}} \times \left[\dfrac{1}{2}(\ln x)^{-1/2} \times \dfrac{1}{x}\right] = \dfrac{1}{2\sqrt{\ln a} \times x \times \sqrt{\ln x}}$

$f'(x) = \dfrac{1}{2x\sqrt{\ln a \times \ln x}}$