USING PARTIAL FRACTIONS TO FIND INTEGRALS

You are now going to use this technique to find and evaluate integrals.

Example 14

Find: **(a)** $\int \frac{dx}{(x-2)(x-1)}$ **(b)** $\int \frac{2x-1}{(x+2)(x-3)}dx$

Solution

(a)

Partial fractions: $\frac{1}{(x-2)(x-1)} = \frac{a}{x-2} + \frac{b}{x-1} = \frac{a(x-1)+b(x-2)}{(x-2)(x-1)}$

Equate numerators: $1 \equiv a(x-1) + b(x-2)$

Let $x = 1$: $1 = -b \quad \therefore b = -1$

Let $x = 2$: $a = 1$

Hence: $\int \frac{dx}{(x-2)(x-1)} = \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx$

$= \log_e(x-2) - \log_e(x-1) + C$

$= \log_e\left|\frac{x-2}{x-1}\right| + C$

(b)

Partial fractions: $\frac{2x-1}{(x+2)(x-3)} = \frac{a}{x+2} + \frac{b}{x-3} = \frac{a(x-3)+b(x+2)}{(x+2)(x-3)}$

Equate coefficients: $2 = a + b$ [1]

$-1 = -3a + 2b$ [2]

Rewrite [1]: $b = 2 - a$

Substitute [1] into [2]: $-1 = -3a + 4 - 2a$

$5a = 5 \quad \therefore a = 1$

Substitute into [1]: $b = 1$

Hence: $\int \frac{2x-1}{(x+2)(x-3)} dx = \int \left(\frac{1}{x+2} + \frac{1}{x-3}\right) dx$

$= \log_e(x+2) + \log_e(x-3) + C$

$= \log_e(x+2)(x-3) + C \quad (x > 3,\ x < -1)$

$= \log_e|(x+2)(x-3)| + C$

Example 15

Find: **(a)** $\int \frac{1-2x}{(x+2)(x^2+1)}dx$ **(b)** $\int \frac{x^2+4x}{(x-1)(x^2+4)}dx$

Solution

(a) Partial fractions:

$$\frac{1-2x}{(x+2)(x^2+1)} = \frac{a}{x+2} + \frac{bx+c}{x^2+1} = \frac{a(x^2+1)+(bx+c)(x+2)}{(x+2)(x^2+1)}$$

$$= \frac{ax^2+a+bx^2+2bx+cx+2c}{(x+2)(x^2+1)}$$

$$= \frac{(a+b)x^2+(2b+c)x+a+2c}{(x+2)(x^2+1)}$$

Equate coefficients:

$0 = a+b$ [1]

$-2 = 2b+c$ [2]

$1 = a+2c$ [3]

Rewrite [1] and [2]: $b=-a, \quad c=-2-2b$

$\therefore$ $c = -2+2a$

Substitute into [3]: $1 = a-4+4a$

$a = 1 \qquad \therefore b=-1, \quad c=0$

Hence:

$$\int \frac{1-2x}{(x+2)(x^2+1)}dx = \int\left(\frac{1}{x+2} - \frac{x}{x^2+1}\right)dx$$

$$= \log_e(x+2) - \frac{1}{2}\log_e(x^2+1) + C$$

(b) Partial fractions:

$$\frac{x^2+4x}{(x-1)(x^2+4)} = \frac{a}{x-1} + \frac{bx+c}{x^2+4} = \frac{a(x^2+4)+(bx+c)(x-1)}{(x-1)(x^2+4)}$$

$$= \frac{(a+b)x^2+(c-b)x+4a-c}{(x-1)(x^2+4)}$$

Equate coefficients:

$1 = a+b$ [1]

$4 = c-b$ [2]

$0 = 4a-c$ [3]

Rewrite [2] and [3]: $b = c-4, c = 4a$

$\therefore$ $b = 4a-4$

Substitute into [1]: $1 = a+4a-4$

$a = 1 \qquad \therefore b=0, \quad c=4$

Hence:

$$\int \frac{x^2+4x}{(x-1)(x^2+4)}dx = \int\left(\frac{1}{x-1} + \frac{4}{x^2+4}\right)dx$$

$$= \log_e(x-1) + 2\tan^{-1}\frac{x}{2} + C$$

Example 21

Find: $\int \frac{d\theta}{\cos\theta}$

Solution

Let $t = \tan\frac{\theta}{2}$ so that $\cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2}{1+t^2}dt$.

Hence:
$$\int \frac{d\theta}{\cos\theta} = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\,dt$$
$$= \int \frac{2}{1-t^2}\,dt$$

Partial fractions:
$$= \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$$
$$= \log_e|1+t| - \log_e|1-t| + C$$
$$= \log_e\left|\frac{1+t}{1-t}\right| + C$$
$$= \log_e\left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right| + C$$

Example 22

Find $\int \frac{dx}{\sin 2x + \cos 2x}$ using the substitution $t = \tan x$.

Solution

$\sin 2x = \frac{2t}{1+t^2}, \quad \cos 2x = \frac{1-t^2}{1+t^2}$

You must recalculate the expression for dx, as $t = \tan x$ not $\tan \frac{x}{2}$:

$$\frac{dt}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + t^2$$

Hence $dx = \frac{dt}{1+t^2}$:

$$\begin{aligned}
\int \frac{dx}{\sin 2x + \cos 2x} &= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2} \\
&= \int \frac{dt}{1+2t-t^2} \\
&= \int \frac{dt}{2-(t^2-2t+1)} \\
&= \int \frac{dt}{2-(t-1)^2} \\
&= \int \frac{dt}{\left(\sqrt{2}-(t-1)\right)\left(\sqrt{2}+(t-1)\right)} \\
&= \frac{1}{2\sqrt{2}} \int \left(\frac{1}{\left(\sqrt{2}+1-t\right)} + \frac{1}{\left(\sqrt{2}-1+t\right)} \right) dt \\
&= \frac{1}{2\sqrt{2}} \left(-\log_e \left|\sqrt{2}+1-t\right| + \log_e \left|\sqrt{2}-1+t\right| \right) + C \\
&= \frac{1}{2\sqrt{2}} \log_e \left| \frac{\sqrt{2}-1+t}{\sqrt{2}+1-t} \right| + C \\
&= \frac{1}{2\sqrt{2}} \log_e \left| \frac{\sqrt{2}-1+\tan x}{\sqrt{2}+1-\tan x} \right| + C
\end{aligned}$$

Note: When using a substitution of $t = \tan f(x)$, $f(x) \neq \frac{x}{2}$, differentiate the substitution to find the link between dx and dt.