INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section the derivatives of the inverse trigonometric functions were established. From these, it follows that:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+C, -a<x<a$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\frac{x}{a}+C, -a<x<a$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C, \text{ for all } x$$

Example 25

Find: (a) $\int \frac{1}{\sqrt{4-x^2}}dx$ (b) $\int \frac{-1}{\sqrt{9-x^2}}dx$ (c) $\int \frac{1}{4+x^2}dx$ (d) $\int \frac{2}{\sqrt{3-x^2}}dx$

Solution

(a) $\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\frac{x}{2}+C$ (as $a=2$)

(b) $\int \frac{-1}{\sqrt{9-x^2}}dx = \cos^{-1}\frac{x}{3}+C$ (as $a=3$)

(c) $\int \frac{1}{4+x^2}dx = \frac{1}{2}\tan^{-1}\frac{x}{2}+C$ (as $a=2$)

(d) $\int \frac{2}{\sqrt{3-x^2}}dx = 2\sin^{-1}\frac{x}{\sqrt{3}}+C$ (as $a=\sqrt{3}$)

Sometimes a slight adjustment is necessary to make the coefficient of x^2 equal to 1.

Example 26

Find: (a) $\int \frac{1}{\sqrt{1-9x^2}}dx$ (b) $\int \frac{1}{1+4x^2}dx$ (c) $\int \frac{2}{\sqrt{4-25x^2}}dx$

Solution

(a) $\int \frac{1}{\sqrt{1-9x^2}}dx = \int \frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}dx = \frac{1}{3}\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}dx = \frac{1}{3}\sin^{-1}3x+C \quad \left(\text{as } a=\frac{1}{3}\right)$

(b) $\int \frac{1}{1+4x^2}dx = \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}dx = \frac{1}{4}\times 2\tan^{-1}2x+C = \frac{1}{2}\tan^{-1}2x+C \quad \left(\text{as } a=\frac{1}{2}\right)$

(c) $\int \frac{2}{\sqrt{4-25x^2}}dx = 2\int \frac{1}{\sqrt{25\left(\frac{4}{25}-x^2\right)}}dx = \frac{2}{5}\int \frac{1}{\sqrt{\frac{4}{25}-x^2}}dx = \frac{2}{5}\sin^{-1}\frac{5x}{2}+C \quad \left(\text{as } a=\frac{2}{5}\right)$

Example 27

Evaluate: **(a)** $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}}$ **(b)** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx$ **(c)** $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **(d)** $\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$

Solution

(a)
$$\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \left[\sin^{-1}\frac{x}{5}\right]_0^{2.5}$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$
$$= \frac{\pi}{6} - 0$$
$$= \frac{\pi}{6}$$

(b)
$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx = \left[\cos^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}}$$
$$= \cos^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{6} - \frac{5\pi}{6}$$
$$= -\frac{2\pi}{3}$$

(c)
$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_1^{\sqrt{3}}$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1}1$$
$$= \frac{\pi}{3} - \frac{\pi}{4}$$
$$= \frac{\pi}{12}$$

(d) As
$$\sqrt{1-4x^2} = \sqrt{4\left(\frac{1}{4}-x^2\right)}$$
$$= 2\sqrt{\frac{1}{4}-x^2}$$
$$\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{4}-x^2}}$$
$$= \frac{1}{2}\left[\sin^{-1}2x\right]_{-\frac{1}{4}}^{\frac{1}{4}}$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{12} + \frac{\pi}{12}$$
$$= \frac{\pi}{6}$$

Example 28

Evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

Solution

There is no standard derivative that yields $\sin^{-1} x$, so finding its primitive is beyond the scope of this Mathematics Extension 1 course. (However, the primitive can be found using the method of 'integration by parts' as studied in the Mathematics Extension 2 course.)

It is possible to evaluate this integral by an alternative method. You can regard the integral as an area and use your knowledge of inverse functions to determine this area.

The graph of $f(x) = \sin^{-1} x$ is shown in the left diagram below. The shaded area A is the area bounded by the graph, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$. This area is equal to $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

The graph of $g(x) = \sin x$ is shown at right, with the reflected area A now enclosed by this graph, $y = \frac{1}{2}$ and the y-axis. The shaded area B on the diagram can be determined by integration, and so:

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sin^{-1} x\,dx &= \text{Area of } A \\
&= \text{Area of rectangle (i.e. } A + B) - \text{Area of } B \\
&= \frac{\pi}{6} \times \frac{1}{2} - \int_0^{\frac{\pi}{6}} \sin x\,dx \\
&= \frac{\pi}{12} - \left[-\cos x\right]_0^{\frac{\pi}{6}} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1
\end{aligned}$$