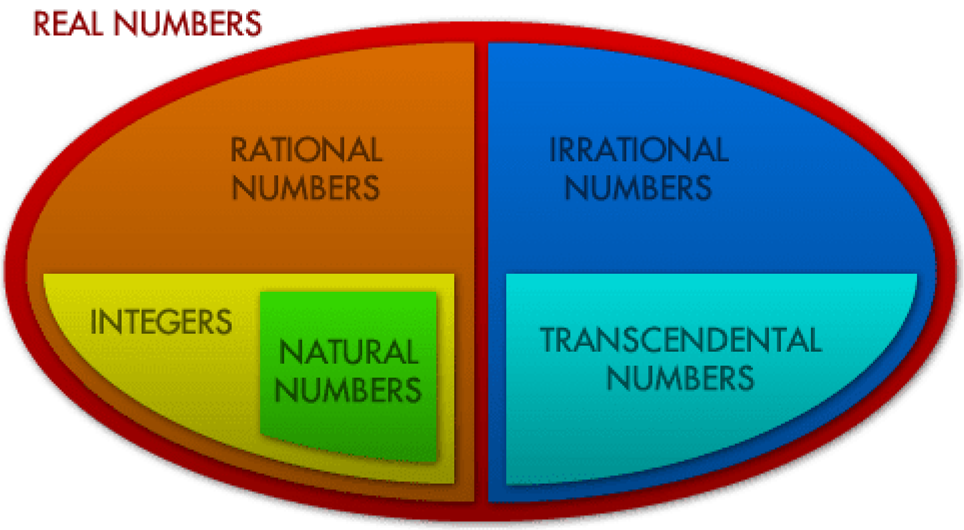
INTRODUCTION TO COMPLEX NUMBERS

- The set of "**natural**" numbers $1, 2, 3, \ldots$ is noted $\mathbb{N}$.
- The set of integers $-1, -2, -3, 0, 1, 2, 3, 4, \ldots$ is noted $\mathbb{Z}$. So the set $\mathbb{N}$ is included in the set $\mathbb{Z}$, which is noted $\mathbb{N} \subset \mathbb{Z}$ (the symbol $\subset$ means "is included within").
- The set of **rational** numbers is noted $\mathbb{Q}$. So the set $\mathbb{Z}$ is included in the set $\mathbb{Q}$, which is noted $\mathbb{Z} \subset \mathbb{Q}$
- **Irrational** numbers are such as $\sqrt{2}, \pi$ and e. The set of rational and irrational numbers together is noted $\mathbb{R}$. So the set $\mathbb{Q}$ is included in the set $\mathbb{R}$, which is noted $\mathbb{Q} \subset \mathbb{R}$.

These relationships can be summarised in the diagram below:



- **Transcendental** numbers are irrational numbers that are not solutions of polynomial equations: for example π and e are transcendental numbers because it has been shown these cannot be solutions of polynomial equations, whereas $\sqrt{2}$ can be solution of a polynomial equation (such as $x^2 = 2$) and therefore is not a transcendental number, but is an irrational number.

THE NEED FOR COMPLEX NUMBERS

Geometric methods were used to solve quadratic equations in Babylonia, Egypt, Greece, China, and India. The Egyptian Berlin Papyrus, dating back to the Middle Kingdom (2050 BC to 1650 BC) contains the solution to a two-term quadratic equation. In 628 AD, Brahmagupta, an Indian mathematician, gave the first explicit - although still not completely general- solution of the quadratic equation $ax^2 + bx + c = 0$.

To solve a quadratic equation: firstly, we calculate the discriminant: $\Delta = b^2 - 4ac$

If $\Delta < 0$, there are no real solutions.

If $\Delta = 0$, there is one real solution $\quad x = \frac{-b}{2a}$

If $\Delta > 0$, there are two real solutions $\quad x = \frac{-b \pm \sqrt{\Delta}}{2a}$

In the 16th century, the focus had turned to solving cubic equations, i.e. $ax^3 + bx^2 + cx + d = 0$
This was solved in 1545 by the Italian mathematician Gerolamo Cardano:
Firstly he made the change of variable $\quad x = t - \frac{b}{3a}$
That changes the cubic equation to the equation $t^3 + pt + q = 0$ which is called a "*depressed*" cubic (as there is no quadratic term), with $p = \frac{3ac - b^2}{3a^2}$ and $q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3}$

Then Cardano proved that this equation has at least one solution[1] which is:

$$t_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Particular case for $x^3 - x = 0$

Obvious solutions are 0, 1 and (-1), but we are going to apply Cardano's formula above to check we can indeed find one of those three.

$$x = t - \frac{b}{3a} = t$$

$$p = \frac{3ac - b^2}{3a^2} = \frac{3 \times 1 \times (-1) - 0^2}{3 \times 1^2} = -1$$

$$q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3} = \frac{2 \times 0^3 - 9a \times 0 \times c + 27a^2 \times 0}{27 \times 1^3} = 0$$

$$t_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = \sqrt[3]{-\frac{0}{2} + \sqrt{\frac{0^2}{4} + \frac{(-1)^3}{27}}} + \sqrt[3]{-\frac{0}{2} - \sqrt{\frac{0^2}{4} + \frac{(-1)^3}{27}}}$$

$$t_1 = \sqrt[3]{\sqrt{\frac{(-1)^3}{27}}} + \sqrt[3]{-\sqrt{\frac{(-1)^3}{27}}} = \sqrt[3]{\sqrt{\frac{-1}{27}}} + \sqrt[3]{-\sqrt{\frac{-1}{27}}} = \sqrt[3]{\frac{\sqrt{-1}}{\sqrt{27}}} + \sqrt[3]{-\frac{\sqrt{-1}}{\sqrt{27}}}$$

Here we note we have a term $\sqrt{-1}$ which seems like nonsense, but we push ahead anyway to check if the formula would still apply.

$$t_1 = \sqrt[3]{\frac{\sqrt{-1}}{3^{\frac{3}{2}}}} + \sqrt[3]{-\frac{\sqrt{-1}}{3^{\frac{3}{2}}}} = \frac{\sqrt[3]{\sqrt{-1}}}{\sqrt[3]{3^{\frac{3}{2}}}} + \frac{\sqrt[3]{-\sqrt{-1}}}{\sqrt[3]{3^{\frac{3}{2}}}} = \frac{\sqrt[3]{\sqrt{-1}}}{3^{\frac{1}{2}}} + \frac{\sqrt[3]{-\sqrt{-1}}}{3^{\frac{1}{2}}} = \frac{\sqrt[3]{\sqrt{-1}}}{\sqrt{3}} + \frac{\sqrt[3]{-\sqrt{-1}}}{\sqrt{3}}$$

$$t_1 = \frac{1}{\sqrt{3}}\left[\left(\sqrt{-1}\right)^{\frac{1}{3}} + \left(-\sqrt{-1}\right)^{\frac{1}{3}}\right]$$

Pretend usual rules of algebra hold for expressions involving $i = \sqrt{-1}$

$$t_1 = \frac{1}{\sqrt{3}}\left[(i)^{\frac{1}{3}} + (-i)^{\frac{1}{3}}\right]$$

And also assume that the algebraic rule $\left(\sqrt{a}\right)^2 = a$ hold for (-1), so $\left(\sqrt{-1}\right)^2 = -1$ (i.e. $i^2 = -1$)

And therefore $i^2 \times i = -1 \times i$ or $i^3 = -i$ or $i = -i^3 = (-1)^3 \times i^3 = (-i)^3$

Therefore:

$$t_1 = \frac{1}{\sqrt{3}}\left[((-i)^3)^{\frac{1}{3}} + (i^3)^{\frac{1}{3}}\right]$$

$$t_1 = \frac{1}{\sqrt{3}}[(-i) + i]$$

$$\boldsymbol{t_1 = 0}$$

Zero is indeed a solution of the equation $x^3 - x = 0$.

So using the "*imaginary*" number $i = \sqrt{-1}$ allowed us in that case to solve this cubic equation.[2]

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[1] https://en.wikipedia.org/wiki/Cubic_equation#Derivation_of_the_roots

[2] In fact it was later proven rigorously that the use of numbers involving the term $\sqrt{-1}$ (later named "*complex*" numbers) is unavoidable when using Cardano's formula for cubic equations with real roots (Confalonieri, Sara (2015). The Unattainable Attempt to Avoid the Casus Irreducibilis for Cubic Equations: Gerolamo Cardano's De Regula Aliza. Springer. pp. 15–16 (note 26). ISBN 3658092750. "*It has been proved that imaginary*

INTRODUCTION TO COMPLEX NUMBERS

This is how complex numbers were introduced. In fact, these complex numbers proved to be extremely useful in plenty of areas of Mathematics and Physics.

DEFINITION OF COMPLEX NUMBERS

Any number z of the form $x + iy$ where x and y are real numbers, is called a **complex** number.

for example, the complex number $(2 + 3i)$

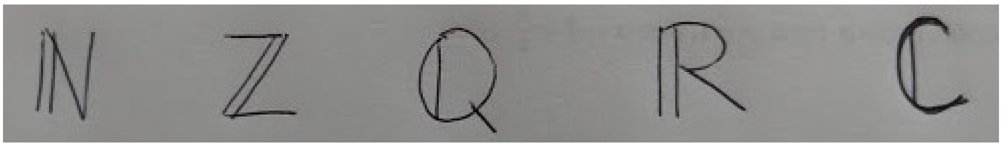
x is called the **real part** of z, denoted by $Re(z) = x$

y is called the **imaginary part** of z, denoted $Im(z) = y$

If the imaginary part of z is zero, i.e. $y = 0$ then z is purely real. This means that the set of real numbers (noted $\mathbb{R}$, as mentioned above) is a subset of the set of complex numbers (which is noted $\mathbb{C}$); i.e. $\mathbb{R} \subset \mathbb{C}$

If the real part of z is zero, i.e. $x = 0$ then z is purely imaginary, e.g. $3i$

The set of **complex numbers,** which includes all real numbers and numbers with an imaginary component, is noted $\mathbb{C}$



numbers have necessarily to appear in the cubic formula when the equation has three real, different roots by Pierre Laurent Wantzel in 1843, Vincenzo Mollame in 1890, Otto Hölder in 1891 and Adolf Kneser in 1892. Paolo Ruffini also provided an incomplete proof in 1799.")

INTRODUCTION TO COMPLEX NUMBERS

USING COMPLEX NUMBERS TO SOLVE QUADRATIC EQUATIONS WHEN DISCRIMINANT Δ IS NEGATIVE

Quadratic equations for which $\Delta < 0$ cannot be solved using real numbers.
The simplest example of such an equation is $x^2 = -1$
However, using the "imaginary" number $i = \sqrt{-1}$ mentioned above, we can state the equation $x^2 = -1$ has two solutions which are:

1) $x = i$ (as $i^2 = -1$) and
2) $x = -i$ (as $(-i)^2 = (-i) \times (-i) = (-1) \times i \times (-1) \times i = (-1) \times (-1) \times i \times i = 1 \times i^2 = -1$)

So the expression $(x^2 + 1)$ can be factorised as $(x + i)(x - i)$

Example 1

Solve the quadratic equation $x^2 - 4x + 13 = 0$.

Solution

Note that the discriminant $\Delta = b^2 - 4ac = 16 - 52 = -36$. Hence the quadratic equation has no real roots and the parabola $y = x^2 - 4x + 13$ is entirely above the x-axis.

However, you can find solutions using **complex numbers**, which are of the form $a + bi$ where a and b are real numbers.

Method 1 Using the quadratic formula

$$x^2 - 4x + 13 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{4 \pm \sqrt{-36}}{2}$$
$$x = 2 \pm 3\sqrt{-1}$$
$$x = 2 \pm 3i$$

Method 2 Completing the square

$$x^2 - 4x + 13 = 0$$
$$x^2 - 4x + 4 + 9 = 0$$
$$(x - 2)^2 = -9$$
$$x - 2 = \pm 3i$$
$$x = 2 \pm 3i$$

WORKING WITH COMPLEX NUMBERS

Equality of complex numbers

Two complex numbers are equal if and only if their real parts and their imaginary parts are equal:

$a + bi = c + di$ if and only if $a = c$ and $b = d$

The "if and only if" in the above statement means that the statement applies forwards and backwards, in other words it is two statements in one:

if $a + bi = c + di$ then $a = c$ and $b = d$

if $a = c$ and $b = d$ then $a + bi = c + di$

Addition of two complex numbers

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ then $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

Example: if $z_1 = 3 + 2i$ and $z_2 = -1 - 5i$ then $z_1 + z_2 = (3 - 1) + (2 - 5)i = 2 - 3i$

Subtraction of two complex numbers

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ then $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

INTRODUCTION TO COMPLEX NUMBERS

Multiplication of two complex numbers

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ then $z_1 \times z_2 = (x_1 + i\,y_1)(x_2 + i\,y_2)$

$z_1 \times z_2 = x_1x_2 + i\,x_1y_2 + i\,y_1x_2 + i^2\,y_1y_2$

$z_1 \times z_2 = x_1x_2 + i\,(x_1y_2 + y_1x_2) - y_1y_2$

$z_1 \times z_2 = x_1x_2 - y_1y_2 + i\,(x_1y_2 + y_1x_2)$

So $Re(z_1z_2) = x_1x_2 - y_1y_2$ and $Im(z) = x_1y_2 + y_1x_2$

The conjugate of a complex numbers

If $z = x + iy$, the **conjugate** of z is defined as $x - iy$ (this is similar to the conjugate of a surd).

It is noted $\bar{z}$. So $\bar{z} = x - iy$

Note that $z\bar{z} = (x + iy)(x - iy) = x^2 - i\,xy + i\,yx - i^2y^2 = x^2 - (-1)y^2 = x^2 + y^2$

$$\text{Therefore} \quad z\bar{z} = x^2 + y^2$$

The product of a complex number and its conjugate is a real number.

Division of two complex numbers

To divide the complex number $z_1 = x_1 + i\,y_1$ by the complex number $z_2 = x_2 + i\,y_2$, we multiply numerator and denominator by the conjugate of z_2.

This has the effect of making the denominator real ("realises the denominator")

This is similar to how you rationalise a denominator when dividing surds.

Example 2

If $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, find:

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 \times z_2$ (d) $z_2\bar{z}_2$ (e) $z_1^{\,2}$ (f) $z_1 \div z_2$

Solution

(a) $z_1 + z_2 = 2 + 3i + (-1 + 4i) = 1 + 7i$

(b) $z_1 - z_2 = 2 + 3i - (-1 + 4i) = 3 - i$

(c) $z_1 \times z_2 = (2 + 3i)(-1 + 4i) = -2 + 8i - 3i + 12i^2 = -2 + 5i - 12 = -14 + 5i$

(d) $z_2\bar{z}_2 = (-1 + 4i)(-1 - 4i) = (-1)^2 - 16i^2 = 1 + 16 = 17$

(e) $z_1^{\,2} = (2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$

(f) $\dfrac{z_1}{z_2} = \dfrac{2 + 3i}{-1 + 4i} = \dfrac{(2 + 3i)}{(-1 + 4i)} \times \dfrac{(-1 - 4i)}{(-1 - 4i)} = \dfrac{-2 - 8i - 3i - 12i^2}{1 - 16i^2} = \dfrac{-2 - 11i + 12}{1 + 16} = \dfrac{10 - 11i}{17} = \dfrac{10}{17} - \dfrac{11}{17}i$

Example 4

Express $z^3 + 64$ as the product of three linear factors. Hence find the three cube roots of -64.

Solution

$$\begin{aligned} z^3 + 64 &= (z+4)(z^2 - 4z + 16) && \text{(sum of two cubes)} \\ &= (z+4)(z^2 - 4z + 4 + 12) && \text{(complete the square)} \\ &= (z+4)((z-2)^2 - 12i^2) && \text{(construct the difference of two squares)} \\ &= (z+4)((z-2)^2 - (2\sqrt{3}i)^2) \\ &= (z+4)(z-2-2\sqrt{3}i)(z-2+2\sqrt{3}i) \end{aligned}$$

The cube roots of -64 are obtained from $z+4=0$, $z-2-2\sqrt{3}i=0$, $z-2+2\sqrt{3}i=0$.

$\therefore$ The cube roots are -4, $2-2\sqrt{3}i$ and $2+2\sqrt{3}i$.

Square roots of complex numbers

The general method for finding the square roots of a complex number is shown below.

Example 5

Find the square roots of $3 + 4i$.

Solution

Let $z = x + iy$, where x, y are real, such that $z^2 = 3 + 4i$:

$$(x+iy)^2 = 3 + 4i$$

$$(x^2 - y^2) + 2xyi = 3 + 4i$$

Equating the real and imaginary parts of LHS and RHS:

$x^2 - y^2 = 3 \quad [1] \qquad 2xy = 4 \quad [2]$

From [2], $y = \frac{2}{x}$, then substitute into [1]:

$$x^2 - \frac{4}{x^2} = 3$$

$$x^4 - 3x^2 - 4 = 0$$

$$(x^2 - 4)(x^2 + 1) = 0$$

$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$

But x is real $\therefore$ $x = \pm 2$ are the only solutions.

Substituting this into [2]: $y = \pm 1$

So the square roots of $3 + 4i$ are $2 + i$ and $-2 - i$, which can be written as $\pm(2 + i)$.