# INDUCTION – WHEN STEP 2 WORKS, BUT NOT STEP 1

A single counterexample is enough to disprove a result.

Some proofs by mathematical induction seemed to be correct but turn out to be incorrect.

The most likely reason is that you have failed to prove Step 1, i.e. that the result is true at the beginning (usually for $n = 1$). This means that the assumed result ($n = k$) is false, so when you prove the results true for $n = k + 1$, you are proving it from an incorrect assumption.

## Example 6

Let $S(n)$ be the statement: $n^2 - n$ is an odd integer, for all positive integers $n$.

**(a)** Show that if $S(k)$ is true, then $S(k + 1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any $n$?

### Solution

**(a)** Let $S(k)$ be that $k^2 - k$ is an odd integer.
Statement: $S(k+1)$: $(k+1)^2 - (k+1)$ is an odd integer.
$(k+1)^2 - (k+1)$
$= (k+1)(k+1-1)$
$= k(k+1)$
$= k^2 - k + 2k$
$= S(k) + 2k$
$=$ Odd + Even = Odd
Hence the result is true for $n = k + 1$ if it is true for $n = k$.

**(b)** $S(1)$ is that $1^2 - 1$ is an odd integer and since this result is 0, then $S(1)$ is false.

**(c)** $S(n)$ is never true because when $n$ is even, $n^2 - n$ is even, and when $n$ is odd, $n^2 - n$ is even.

## Example 7

Use mathematical induction to prove that $2^n - 1$ is prime if $n$ is prime.

### Solution

Using a table of values to explore this for some values of $n$:

| $n$ | 2 | 3 | 5 | 7 |
|---|---|---|---|---|
| $2n - 1$ | 3 | 7 | 31 | 127 |

This suggests that the result works, although finding a pattern for the prime numbers for $n$ is not possible.

The next prime values for $n$ are 11, 13, 17, 19, 23, …

Any non-prime value for $2^n - 1$ is enough to say the original statement is false. Testing these values finds for $n = 11$: $2^{11} - 1 = 2047$. As $2047 = 23 \times 89$, this is not prime, so the initial assumption is false.

The result cannot be proved by mathematical induction.